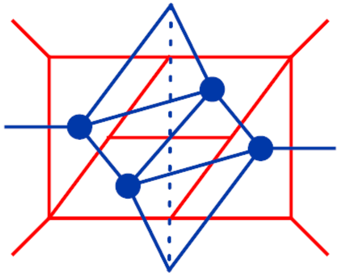
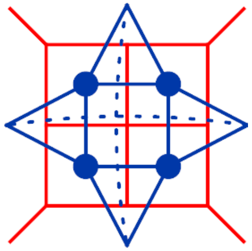
Gluon Scattering in N=4 Super-Yang-Mills Theory from Weak to Strong Coupling





Lance Dixon (CERN & SLAC)

with Z. Bern, J.J Carrasco, M. Czakon, H. Johansson,
D. Kosower, R. Roiban, V. Smirnov,
M. Spradlin, C. Vergu, A. Volovich

Neve Shalom, 12 April 2011

N=4 super-Yang-Mills theory

massless spin 1 gluon

4 massless spin 1/2 gluinos

6 massless spin 0 scalars

all states in adjoint representation, all linked by N=4 supersymmetry

- Interactions uniquely specified by gauge group, say $SU(N_c)$, 1 coupling g
- Exactly scale-invariant (conformal) field theory: $\beta(g) = 0$ for all g

Planar N=4 SYM and AdS/CFT

recent review: Alday, Roiban, 0807.1889

- In the 't Hooft limit, $N_c \rightarrow \infty$

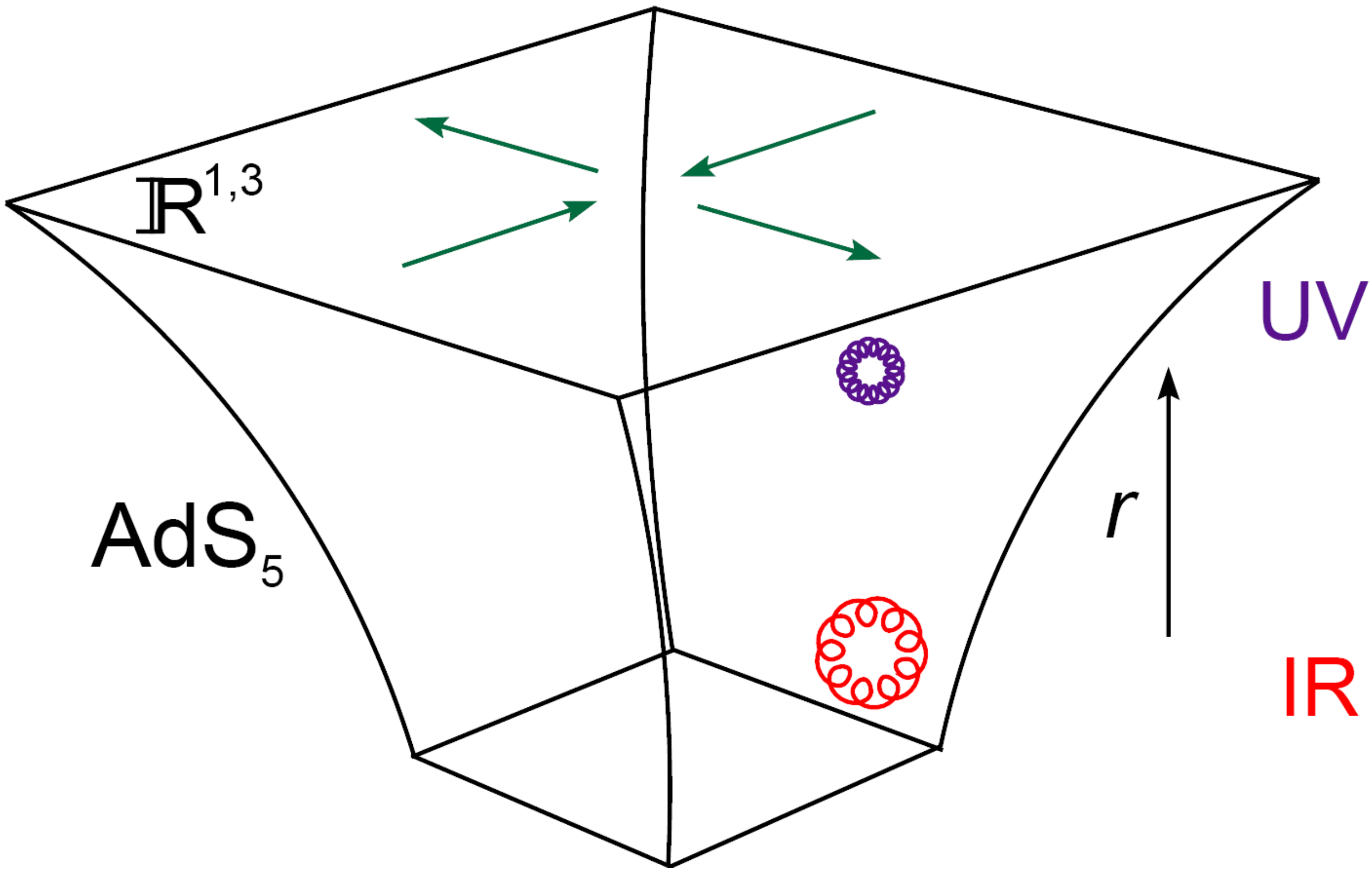
  $\lambda = g^2 N_c$ fixed, planar diagrams dominate
- AdS/CFT duality

  suggests that weak-coupling perturbation series in λ for large-N_c (planar) N=4 SYM should have hidden structure, because

  large λ limit ←→ weakly-coupled gravity/string theory on $AdS_5 \times S^5$

  Maldacena; Gubser, Klebanov, Polyakov; Witten

AdS/CFT in one picture



Three Hidden Structures Recently Unveiled in Planar N=4 SYM

- Exponentiation of finite terms in the amplitude (for 4 and 5 gluons)
- Dual (super)conformal invariance
- Equivalence between (MHV) amplitudes and Wilson lines

Gluon scattering in N=4 SYM

- *gg* → *gg* 4-gluon scattering amplitude not protected by supersymmetry (unlike e.g. anomalous dimensions of BPS states).
- How does series organize itself into simple result, from gravity/string point of view?

Anastasiou, Bern, LD, Kosower, hep-th/0309040

- Can we propose an exact form for the scattering amplitude, which interpolates between weak and strong coupling?
- Cusp anomalous dimension $\gamma_K(\lambda)$ is a new, nontrivial example, solved to all orders in λ using integrability

Beisert, Eden, Staudacher, hep-th/0610251

An Ansatz

- $\gamma_K(\lambda)$ **one of just four functions of λ alone, which fully specify *gg* → *gg* scattering to all orders in λ, for any scattering angle θ (value of t/s).**
- **Also n-gluon (2 → n-2) MHV amplitudes**

$g^-g^- \to g^+g^-g^+g^-g^+g^+$ **?** Bern, LD, Smirnov (2005)

Good evidence in favor of ansatz for n = 4,5, due to symmetry of $AdS_5 \times S^5$:

(fermionic) T-duality

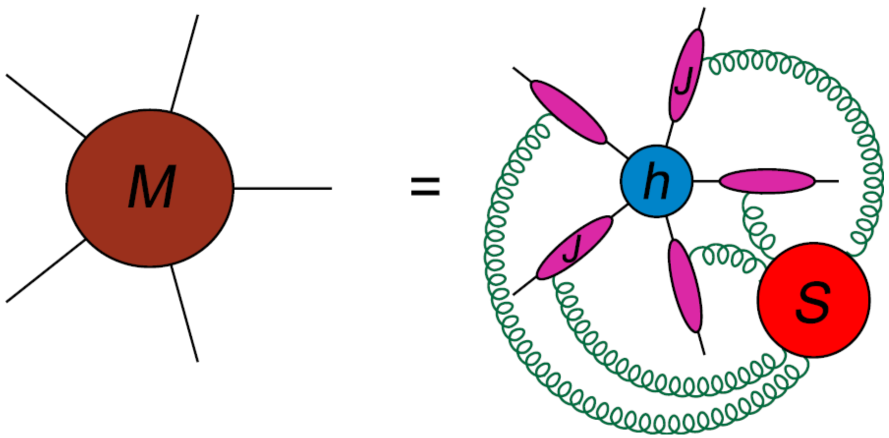
←→ dual (super)conformal invariance

Kallosh, Tseytlin, hep-th/9808088;
Alday, Maldacena, 0705.0303; 0710.1060;
Berkovits, Maldacena, 0807.3196;
Beisert, Ricci, Tseytlin, Wolf, 0807.3228

- It is **wrong** for n > 5. Alday, Maldacena, 0710.1060; Drummond, Henn, Korchemsky, Sokatchev, 0712.4138;
Bartels, Lipatov, Sabio Vera, 0802.2065;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465
- Yet, seeing how it fails has also been very fruitful…

A technical issue…

Beyond tree-level, scattering amplitudes afflicted with infrared (soft & collinear) divergences



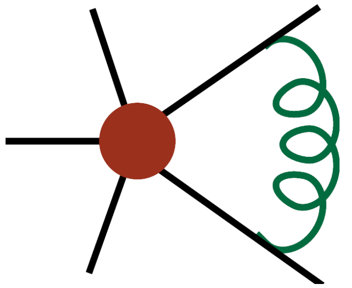
Fortunately we know how to deal with them in gauge theories: Use dimensional regulation, $D = 4-2\varepsilon$. Exploit knowledge gained in QED and QCD studies about soft/collinear factorization & exponentiation

Dimensional Regulation in the IR

One-loop IR divergences are of two types:

Soft

$$\int_0 \frac{d\omega}{\omega} \rightarrow \int_0 \frac{d\omega}{\omega^{1+\epsilon}} \propto \frac{1}{\epsilon}$$

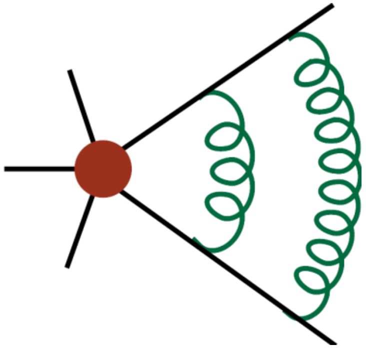


Collinear (with respect to massless emitting line)

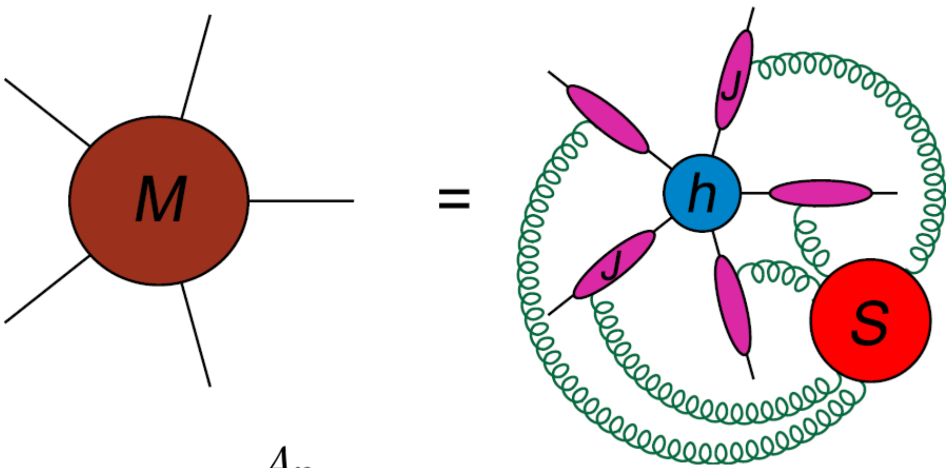
$$\int_0 \frac{dk_T}{k_T} \rightarrow \int_0 \frac{dk_T}{k_T^{1+\epsilon}} \propto \frac{1}{\epsilon}$$

Overlapping soft + collinear divergences imply leading pole is $\frac{1}{\epsilon^2}$ at 1 loop

$\frac{1}{\epsilon^{2L}}$ at L loops



Soft/Collinear Factorization



Magnea, Sterman (1990);
Sterman, Tejeda-Yeomans,
hep-ph/0210130

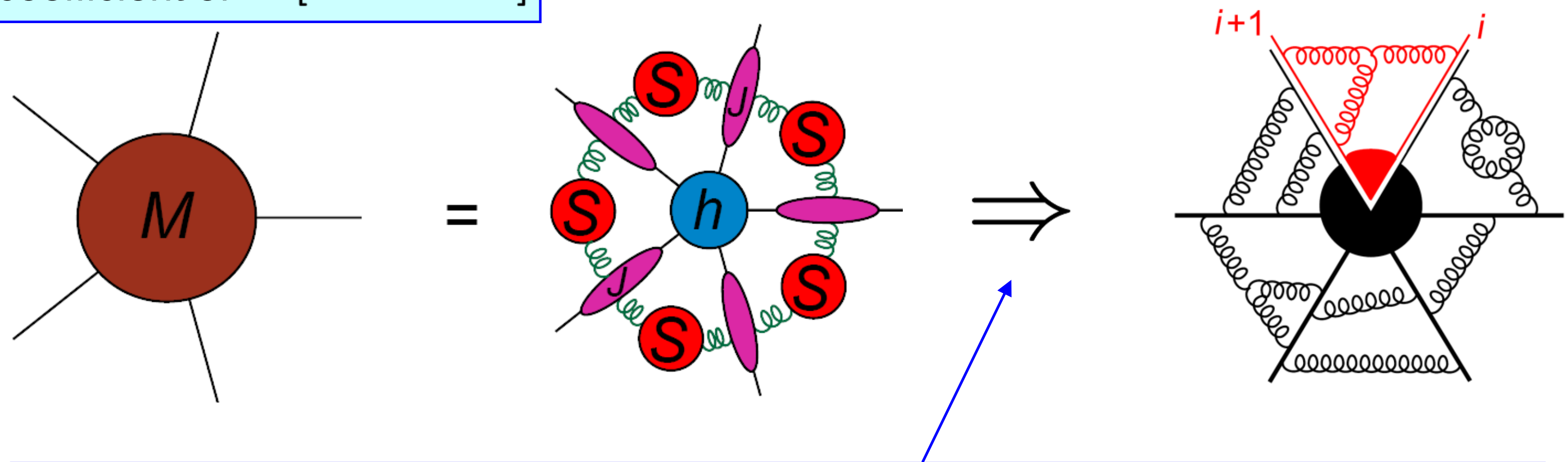
$$\mathcal{M}_n \equiv \frac{\mathcal{A}_n}{\mathcal{A}_n^{\text{tree}}}$$

$$\mathcal{M}_n = S(k_i, \mu, \alpha_s(\mu), \epsilon) \times \left[\prod_{i=1}^{n} J_i(\mu, \alpha_s(\mu), \epsilon) \right] \times h_n(k_i, \mu, \alpha_s(\mu), \epsilon)$$

- *S = soft function (only depends on color of i^{th} particle)*
- *J = jet function (color-diagonal; depends on i^{th} spin)*
- *h_n= hard remainder function (finite as $\epsilon \to 0$)*

Simplification at Large N_c (Planar Case)

coefficient of $\mathrm{Tr}[T^{a_1} \ldots T^{a_n}]$



- *Soft function* only defined up to a multiple of the identity matrix in color space
- Planar limit is color-trivial; can absorb S into J_i
- If all n particles are identical, say gluons, then each "wedge" is the square root of the "$gg \to 1$" process (Sudakov form factor):

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]}\left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} \times h_n\left(k_i, \mu, \alpha_s, \epsilon\right)$$

Exponentiation

Bern, LD, Smirnov, hep-th/0505205

Inspired by IR structure of QCD, Mueller, Collins, Sen, Magnea, Sterman,…
based on evidence collected at 2 and 3 loops for *n=4,5* using **generalized unitarity and factorization**, **ansatz proposed**:

$$\frac{\mathcal{A}_n}{\mathcal{A}_n^{\rm tree}} \equiv \mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty}[\frac{\lambda}{8\pi^2}]^l\Big(f^{(l)}(\epsilon)M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)\Big)\Big]$$

constants, indep.of kinematics

all kinematic dependence in known 1-loop amplitude (normalized by tree)

n=4 $\Rightarrow$

$$\mathcal{M}_4|_{\rm finite} = \exp\Big[\frac{1}{8}\gamma_K(\lambda)\,\ln^2\Big(\frac{s}{t}\Big) + {\rm const.}\Big]$$

Alday Maldacena 0705.0303 0710.1060

Confirmed at strong coupling using AdS/CFT, directly at *n=4*, indirectly at *n=5*. (But: fails for *n > 5*.)

The constants

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon\, f_1^{(l)} + \epsilon^2\, f_2^{(l)}$$

collects **3 series of constants**:

$$f_0^{(l)} = \frac{1}{4}\hat{\gamma}_K^{(l)} \qquad f_1^{(l)} = \frac{l}{2}\hat{\mathcal{G}}_0^{(l)} \qquad f_2^{(l)} = (???) \qquad C^{(l)} = (???)$$

control IR poles

$\hat{\gamma}_K^{(l)}$, $\hat{\mathcal{G}}_0^{(l)}$ are ***l***-loop coefficients of

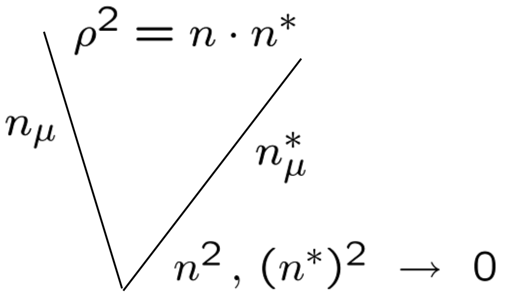
- cusp anomalous dimension $\gamma_K(\lambda)$
 (source term for differential equation for Sudakov form factor)
- "collinear" anomalous dimension $\mathcal{G}_0(\lambda) = G(-1, \lambda, \epsilon = 0)$
(integration constant for differential equation)

Beisert, Eden, Staudacher, hep-th/0610251

Cusp anomalous dimension

VEV of Wilson line with kink or cusp in it obeys renormalization group equation:

$$\left(\rho\frac{\partial}{\partial\rho} + \beta(g)\frac{\partial}{\partial g}\right) \ln W(\rho, g) = -2\,\gamma_K(g)\ \ln \rho^2 + \mathcal{O}(\rho^0)$$



Polyakov (1980); Ivanov, Korchemsky, Radyushkin (1986); Korchemsky, Radyushkin (1987)

Cusp (soft) anomalous dimension $\gamma_K(g)$ controls

large-spin limit of anomalous dimensions γ_j of leading-twist operators with spin *j:* $\bar{q}(\gamma^+ D_+)^j q$

$$\gamma_j = \frac{1}{2}\gamma_K(g)\ \ln j\ +\ \mathcal{O}(j^0)$$

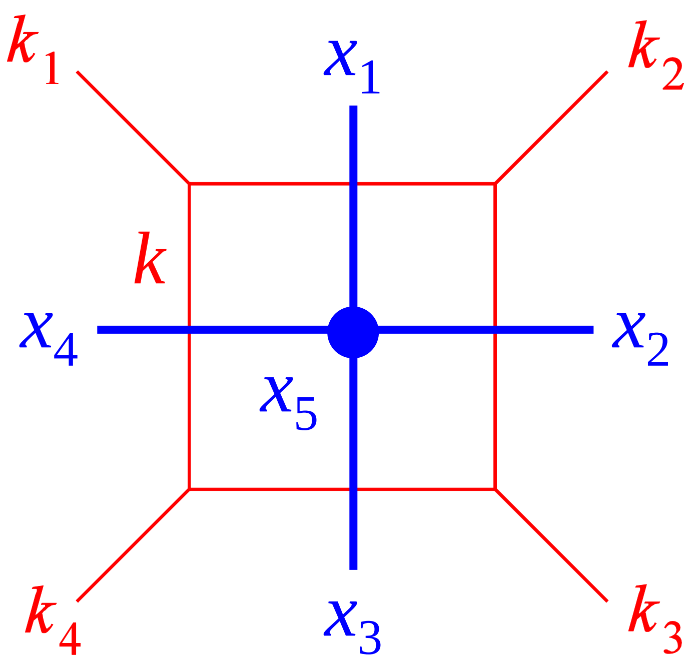
Korchemsky (1989);
Korchemsky, Marchesini (1993)

Dual Conformal Invariance

Broadhurst (1993); Lipatov (1999); Drummond, Henn, Smirnov, Sokatchev, hep-th/0607160

A conformal symmetry acting in momentum space, on dual or sector variables x_i

First seen in N=4 SYM planar amplitudes in the loop integrals



$$I = \int d^4k \frac{(k_1+k_2)^2(k_2+k_3)^2}{k^2(k-k_1)^2(k-k_1-k_2)^2(k+k_4)^2}$$

$$I = \int d^4x_5 \frac{x_{24}^2 x_{13}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

$k_1 = x_{41}$

$k_2 = x_{12}$

$k_3 = x_{23}$

$k_4 = x_{34}$

$k = x_{45}$

invariant under inversion:

$$x_i^\mu \rightarrow \frac{x_i^\mu}{x_i^2}$$

$$x_{ij}^2 \rightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}, \qquad \mathrm{d}^4 x_i \rightarrow \frac{\mathrm{d}^4 x_i}{x_i^8}$$

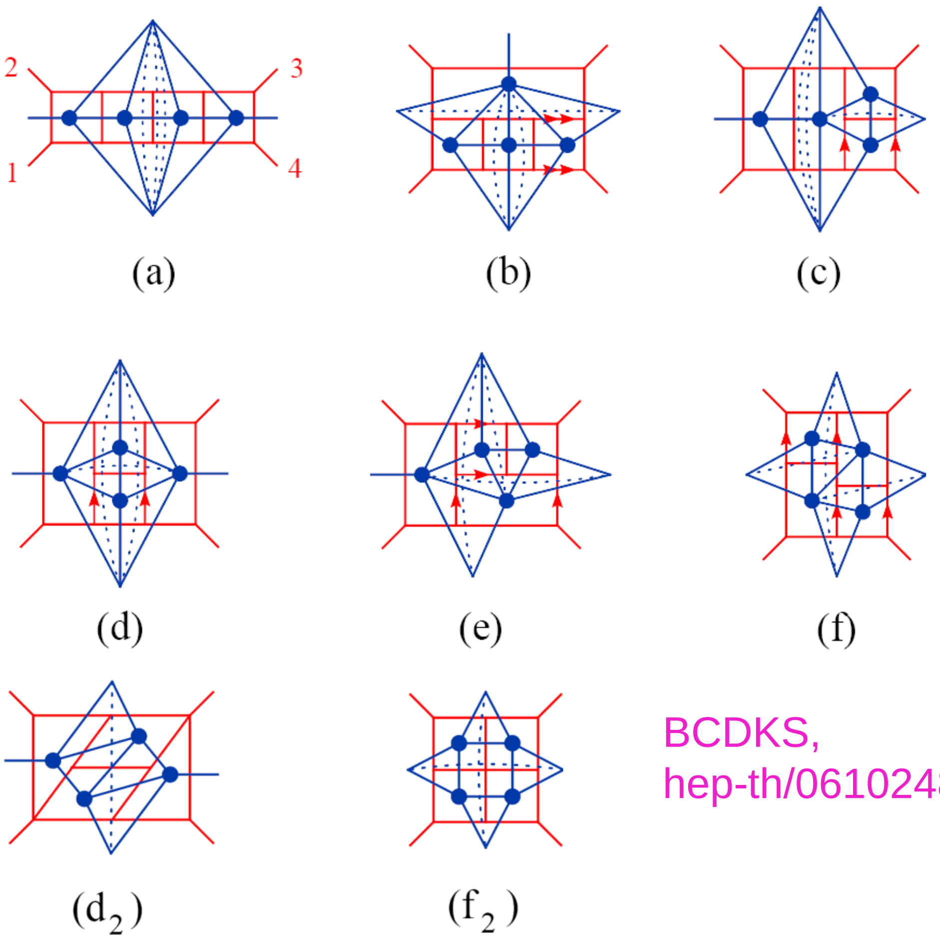
Dual conformal invariance (cont.)

• Simple graphical rules:
4 (net) lines into inner x_i
1 (net) line into outer x_i
• Dotted lines are for numerator factors

4 loop planar integrals all of this form

also true at 5 loops

BCJK, 0705.1864



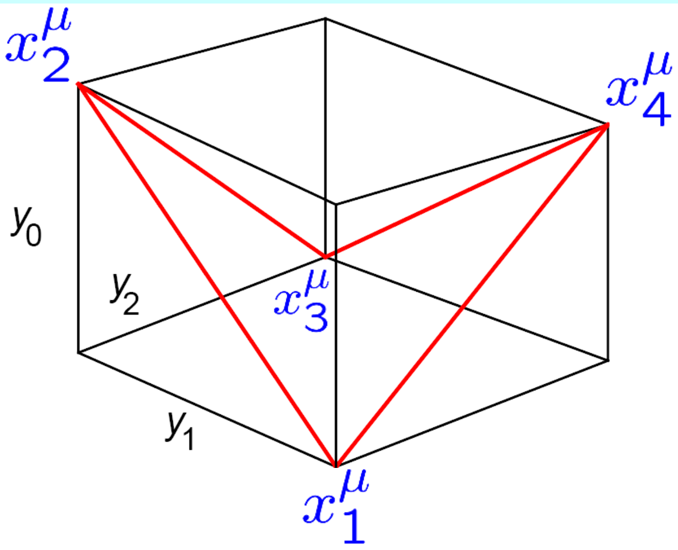
BCDKS, hep-th/0610248

Insight from string theory

- As a property of full amplitudes, rather than integrals, dual conformal invariance follows, at strong coupling, from bosonic T duality symmetry of $AdS_5 \times S^5$.
- Also, strong-coupling calculation ~ equivalent to computation of Wilson line for n-sided polygon with vertices at x_i

Alday, Maldacena, 0705.0303

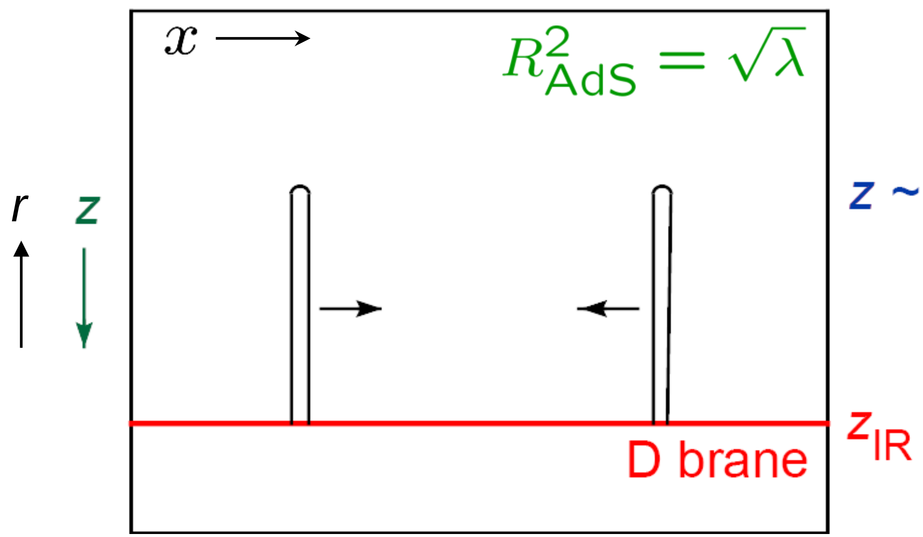
Wilson line blind to helicity formalism
– doesn't know MHV from non-MHV



Scattering at strong coupling

Alday, Maldacena, 0705.0303

- Use AdS/CFT to compute an appropriate scattering amplitude
- High energy scattering in string theory is semi-classical



Gross, Mende (1987,1988)

$z \sim s^{-1/2}, \ \ t^{-1/2}$

$$z \equiv \frac{R_{\rm AdS}^2}{r}$$

Evaluated on the classical solution, action is imaginary → exponentially suppressed tunnelling configuration

$$A_4 \ \sim \ \exp[iS_{\rm cl}] \ \sim \ \exp[-(-iS_{\rm cl})] \ \sim \ \exp[-\sqrt{\lambda}\ln^2(z/z_{\rm IR})]$$

Can use dimensional regularization instead of $z_{\rm IR}$

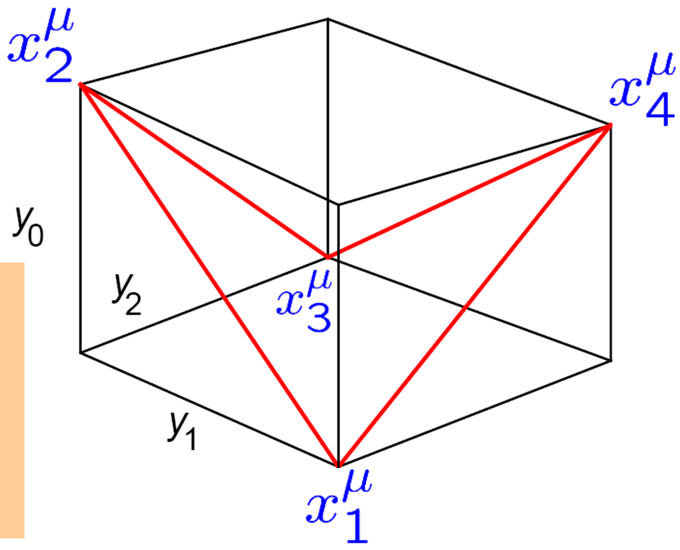
Dual variables and strong coupling

• T-dual momentum variables y^μ introduced

• Boundary values for world-sheet are light-like segments in y^μ:

$\Delta y^\mu = 2\pi k^\mu$ for gluon with momentum k^μ

• For example,
for *gg* → *gg* 90-degree scattering,
s = *t* = -*u*/2, the boundary looks like:

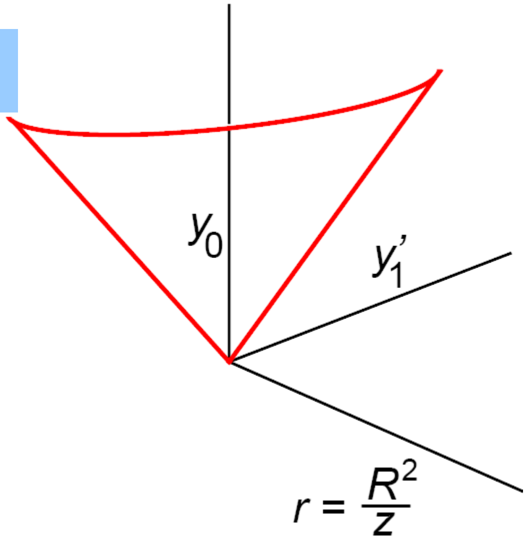


Corners (cusps) located at x_i^μ
– same dual momentum variables introduced above for discussing dual conformal invariance of integrals!!

Cusps in the solution

- Near each corner, solution has a cusp

Kruczenski, hep-th/0210115

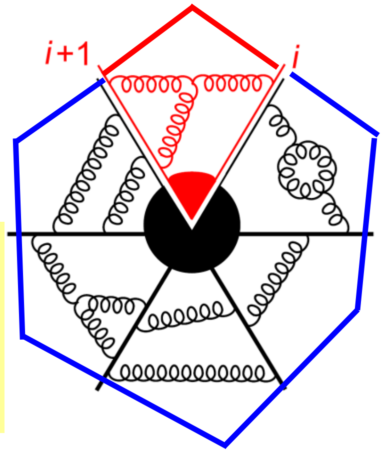


$$r = \sqrt{(2+\epsilon)(y_0^2 - y_1'^2)} \equiv \sqrt{(2+\epsilon)y^+y^-}$$

- Classical action divergence is regulated by ϵ

$$iS = -S_E = -\frac{R^2}{4\pi}\int d\sigma d\tau$$

$$\rightarrow \; -R^2 \int_0 \frac{dy^+dy^-}{(y^+y^-)^{1+\epsilon/2}} \sim -\frac{\sqrt{\lambda}}{\epsilon^2} \sim -\frac{\gamma_K(\lambda)}{\epsilon^2}$$



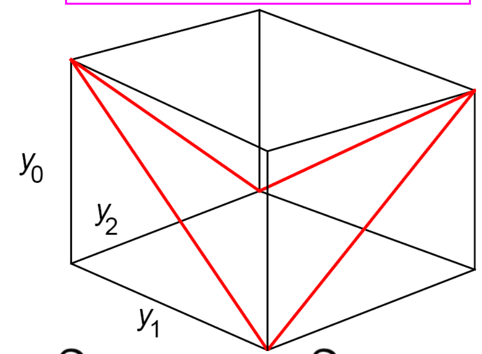
- Cusp in (y,r) is the strong-coupling limit of the red wedge; i.e. the Sudakov form factor.

Buchbinder, 0706.2015

The full solution

- Divergences only come from corners; can set *D=4* in interior.
- Evaluating the action as $\varepsilon \to 0$ gives:

Alday, Maldacena, 0705.0303



$$A_4 = \exp(-S_E)$$

$\gamma_K(\lambda)$ $\qquad \mathcal{G}_0(\lambda)$

$$-S_E = \Big(-\frac{1}{\epsilon^2}\frac{\sqrt{\lambda}}{2\pi} - \frac{1}{\epsilon}\frac{\sqrt{\lambda}}{4\pi}(1-\ln 2)\Big)\Big[\Big(\frac{\mu^2}{-s}\Big)^{\epsilon} + \Big(\frac{\mu^2}{-t}\Big)^{\epsilon}\Big]$$
$$+\frac{\sqrt{\lambda}}{4\pi}\Big[\ln^2\frac{s}{t} + \tilde{C}\Big]$$

combination of $f_2(\lambda) \oplus C(\lambda)$

$\gamma_K(\lambda) \times M_4^{(1)}(s,t)$

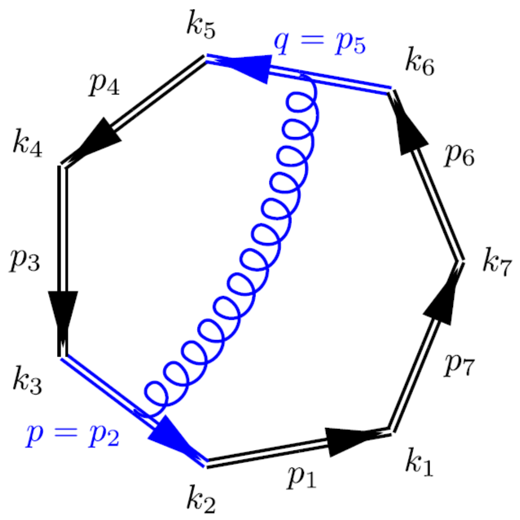
Matches BDS ansatz perfectly (*n=4*)

Dual variables and Wilson lines at weak coupling

• Inspired by Alday, Maldacena, a sequence of computations of Wilson-line configurations with same "dual momentum" boundary conditions:

• One loop, *n=4* — Drummond, Korchemsky, Sokatchev, 0707.0243

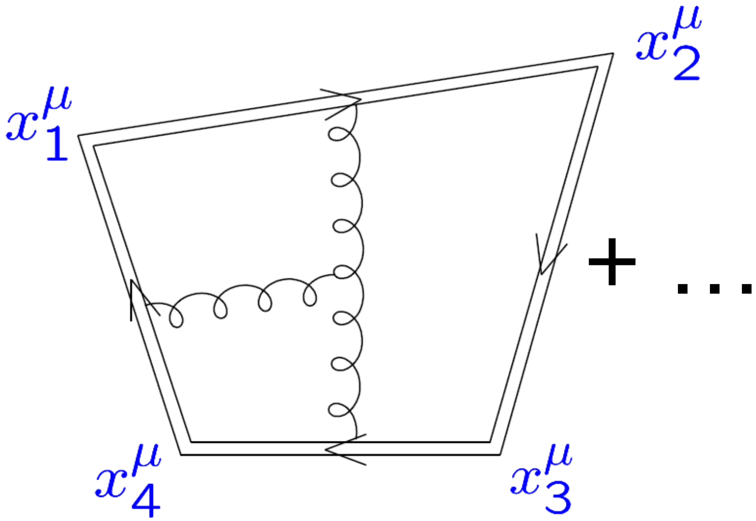
• One loop, any *n* — Brandhuber, Heslop, Travaglini, 0707.1153



Dual variables and Wilson lines at weak coupling (cont.)

- Two loops, *n=4,5*

Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223



- In every case, Wilson-line **matches** full scattering amplitude [MHV amplitude for *n>5*] (!) – up to **additive constants**, e.g. $\mathcal{G}_0(\lambda) \neq \mathcal{G}_{\rm eik}(\lambda)$

Wilson lines obey an "anomalous" (due to IR divergences) dual conformal Ward identity – totally fixes their structure for *n=4,5*.

DHKS, 0712.1223

Dual (super)conformal invariance

• Surprisingly, dual conformal invariance and Wilson line equivalence both persist to weak coupling for MHV amp's

Drummond, Korchemsky, Sokatchev, 0707.0243;
Brandhuber, Heslop, Travaglini, 0707.1153; } 1 loop

Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223 2 loops

• Can embed dual conformal invariance into a richer dual superconformal invariance (needed to understand structure of non-MHV amplitudes)

DHKS, 0807.1095, 0808.0491

Whole structure now explained better, as image of superconformal invariance under a combined bosonic and fermionic T duality symmetry

Berkovits, Maldacena, 0807.3196; Beisert, Ricci, Tseytlin, Wolf, 0807.3228

More than 4 gluons

- Ansatz known to work for $n = 5$ (all MHV) two loops

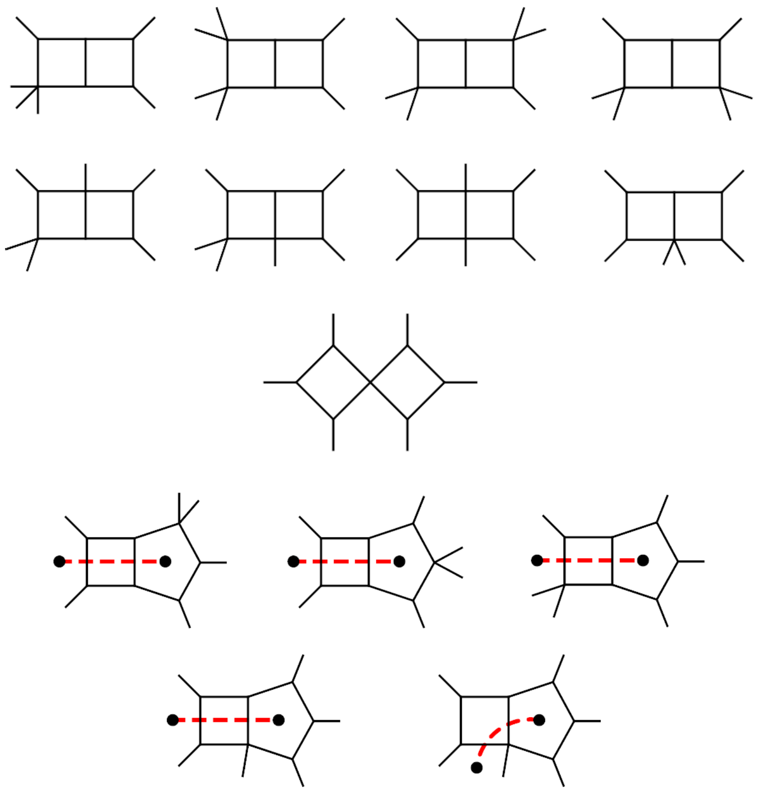
  Cachazo, Spradlin, Volovich, hep-th/0602228;
  Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074

- Should work for $n = 5$ to all loops, assuming dual conformal invariance.
- $n = 6$ is first place it does not fix form of amplitude, due to cross ratios such as $u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$

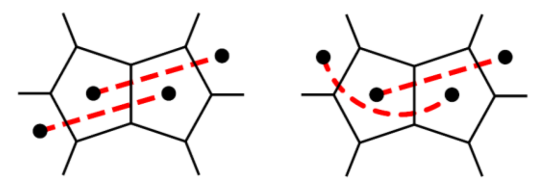
- There were indications of a **failure looming** for $n = 6$, based on:
- A large n, strong-coupling limit — Alday, Maldacena, 0710.1060
- A Wilson line calculation — Drummond, Henn, Korchemsky, Sokatchev, 0712.4138
- A high-energy/Regge limit — Bartels, Lipatov, Sabio Vera, 0802.2065

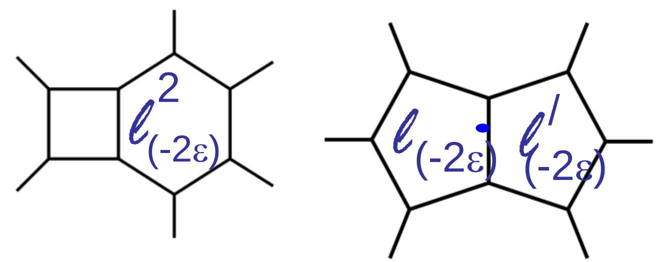
→ Compute (parity-even part of) two-loop 6-point amplitude

Bern, LD, Kosower, R. Roiban, M. Spradlin, C. Vergu, A. Volovich, 0803.1465



all with dual conformal invariant integrands (including prefactors)





Two loop n=6 amplitude

- Find all the correct $1/\varepsilon^4$,…, $1/\varepsilon$ poles.
- $O(\varepsilon^0)$ numerical evaluation confirms that ABDK/BDS ansatz for scattering amplitudes needs correction.

$$A_6^{(2)} = A_6^{(2)\,\mathrm{BDS}} + R_6^{(2)}$$

- Correction term $R_6^{(2)}$ agrees precisely (numerically) with dual conformal invariant Wilson loop function

Drummond, Henn, Korchemsky, Sokatchev, 0712.4138; 0803.1466

- **Parity-odd** part of the amplitude also computed, up to $\ell_{(-2\varepsilon)}$ terms, using leading singularities

Cachazo, Spradlin, Volovich, 0805.4832

also consistent with

"MHV amplitudes = Wilson loops"

What is $R_6^{(2)}$?

- Weight 4 transcendentality function: $\mathrm{Li}_4(\ldots) + \ldots$

- Totally symmetric function of 3 dual conformal ratios:

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}} \qquad u_2 = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} = \frac{s_{23} s_{56}}{s_{234} s_{123}} \qquad u_3 = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2} = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$

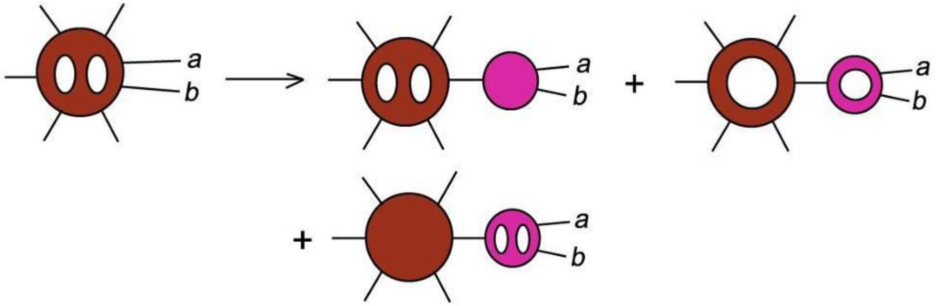
$$R_6^{(2)}(u_1, u_2, u_3) = R_6^{(2)}(u_2, u_3, u_1) = R_6^{(2)}(u_2, u_1, u_3)$$

- Vanishes in 2-particle collinear limits:

$$R_6^{(2)}(u_1, u_2, u_3) \to 0 \qquad u_1 \to 0, \quad u_2 + u_3 \to 1$$

a = 1
b = 2

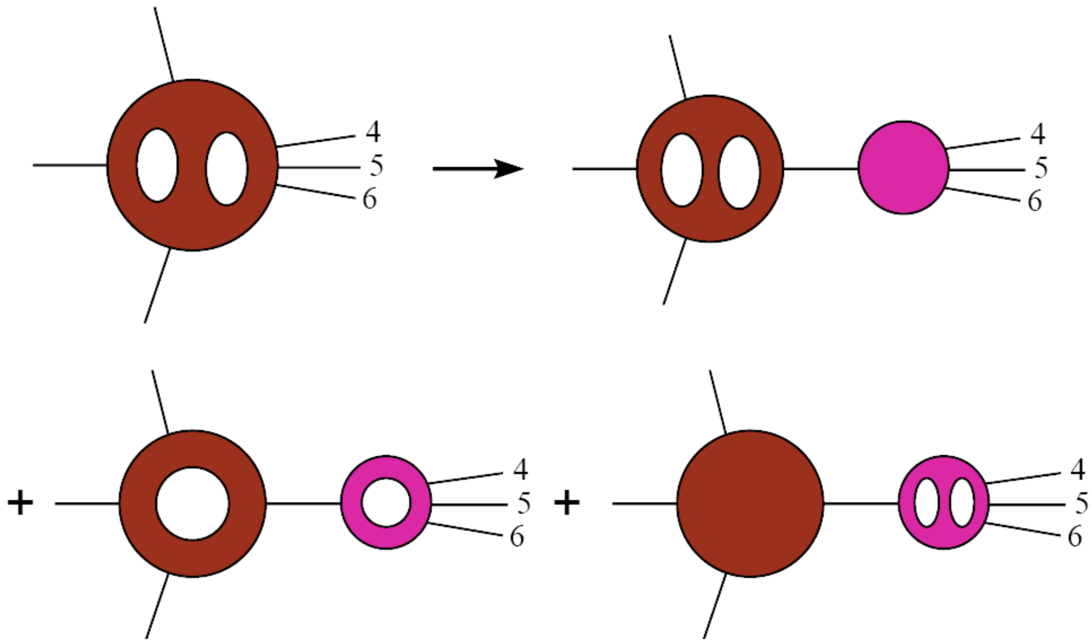
because ansatz was built to be consistent with two-loop collinear behavior:



What is $R_6^{(2)}$? (cont.)

- Nontrivial in 3-particle collinear limits:

$$\bar{u}_1 = \frac{s_{45}}{s_{456}} \frac{1}{1-z_3}, \qquad \bar{u}_2 = \frac{s_{56}}{s_{456}} \frac{1}{1-z_1}, \qquad \bar{u}_3 = \frac{z_1 z_3}{(1-z_1)(1-z_3)}$$



Because all 3 u_i are generic, if we know $R_6^{(2)}$ then we also know this splitting amplitude

$R_6^{(2)}$

• Worked out analytically, using Wilson loop representation, first by Del Duca, Duhr, Smirnov, 0911.5332, 1003.1702
I7 pages of Goncharov polylogarithms.

• Simplified using "symbol" operation (multiple differentiation) by Goncharov, Spradlin, Vergu, Volovich, 1006.5703

$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{\Delta}}{2u_1u_2u_3} \qquad \Delta = (u_1 + u_2 + u_3 - 1)^2 - 4u_1u_2u_3$$

$$R_6^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1 - 1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1 - 1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

$$L_4(x^+, x^-) = \frac{1}{8!!}\log(x^+x^-)^4$$
$$+ \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+) + \ell_{4-m}(x^-))$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n(1/x)\right)$$
$$J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-))$$

New integrand representations

- Based on **"momentum twistors"** – a "super" version of dual variables. Solves constraints from conservation of both momentum and super-momentum Hodges, 0905.1473; Arkani-Hamed et al., 0909.0483
- Factorize momenta into spinors $k_b^\mu (\sigma_\mu)^{\alpha\dot\alpha} = \lambda_b^\alpha \tilde\lambda_b^{\dot\alpha}$
- Add Grassmann variables $\tilde\eta_b^A$ for N=4 supersymmetry Nair, 1988

→ Compute superamplitudes for $\Phi(\tilde\eta)$

$$\Phi(\tilde\eta) = g^+ + \tilde\eta^A \tilde g_A^+ + \frac{1}{2}\tilde\eta^A\tilde\eta^B \phi_{AB} + \frac{1}{6}\tilde\eta^A\tilde\eta^B\tilde\eta^C \epsilon_{ABCD}\tilde g^{D-} + \frac{1}{24}\tilde\eta^A\tilde\eta^B\tilde\eta^C\tilde\eta^D\epsilon_{ABCD}g^-$$

- Now change variables further, $(\tilde\lambda_b^{\dot\alpha}, \tilde\eta_b^A) \rightarrow (\mu_b, \eta_b^A)$

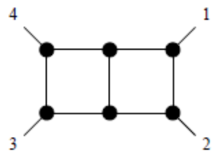
$$\tilde\lambda_b = \frac{\langle b+1\, b\rangle \mu_{b-1} + \langle b-1\, b+1\rangle \mu_b + \langle b\, b-1\rangle \mu_{b+1}}{\langle b-1\, b\rangle\langle b\, b+1\rangle}$$

$$\tilde\eta_b = \frac{\langle b+1\, b\rangle \eta_{b-1} + \langle b-1\, b+1\rangle \eta_b + \langle b\, b-1\rangle \eta_{b+1}}{\langle b-1\, b\rangle\langle b\, b+1\rangle}$$

momentum (super)twistors

$$\mathcal{Z}^D = \begin{pmatrix} \lambda \\ \mu \\ \eta \end{pmatrix}$$

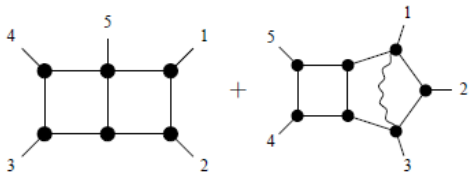
New integrand rep's (cont.)



$+$ cyclic (no repeat)

$\langle 2341\rangle\langle 3412\rangle\langle 4123\rangle$

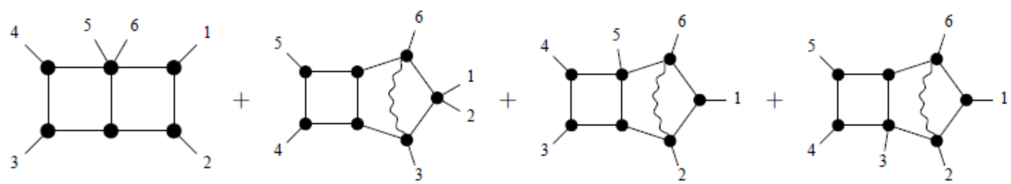
and



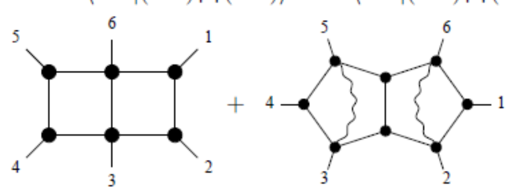
$+$ cyclic (no repeat)

$\langle 2345\rangle\langle 5123\rangle\langle 3412\rangle$ $\quad$ $\langle 3451\rangle\langle 4513\rangle \times\langle AB|(512)\bigcap(234)\rangle$

while the 6-particle amplitude is



$\langle 2345\rangle\langle 6123\rangle\langle 3412\rangle$ $\quad$ $\langle 3456\rangle\langle 4563\rangle \times\langle AB|(561)\bigcap(234)\rangle$ $\quad$ $\langle 2345\rangle\langle 3462\rangle \times\langle AB|(561)\bigcap(123)\rangle$ $\quad$ $\langle 3456\rangle\langle 4562\rangle \times\langle AB|(561)\bigcap(123)\rangle$



$+$ cyclic (no repeat)

$\langle 3456\rangle\langle 6123\rangle\langle 4512\rangle$ $\quad$ $\langle 6235\rangle \times\langle AB|(234)\bigcap(456)\rangle \times\langle CD|(561)\bigcap(123)\rangle$

Numerator factors simplify, number of required integrals is reduced, for n > 5.
Arkani-Hamed et al., 1008.2958

$$\langle a\ b\ c\ d\rangle = \epsilon_{IJKL} Z_a^I Z_b^J Z_c^K Z_d^L$$

bosonic part (λ,μ)

Simpler starting point for evaluating integrals for amplitude
Drummond, Henn, 1008.2965
Drummond, Henn, Trnka, 1010.3679

Conclusions & Open Questions

- Due to dual conformal symmetry, finite terms in planar *gg* → *gg* amplitudes in planar N=4 SYM exponentiate, just like IR divergences. Full amplitude depends on just 4 functions of λ alone (one already "known" to all orders, so n=4 problem (also n=5) is at least "1/4" solved!
- What is AdS/operator interpretation of the other 3 functions? Can one find integral equations for them?
- How are AdS/CFT, integrability, [dual] conformality & Wilson lines related?
- **Why** are MHV amplitudes = Wilson lines ?
- What are the n>6 "remainder" functions in N=4 SYM?
- What happens for non-MHV amplitudes? From form of 1-loop amplitudes, answer must be more complex.

recent work by Eden, Korchemsky, Sokatchev, Maldacena, Mason, Skinner,…

Extra Slides

Leading transcendentality relation between QCD and N=4 SYM

- KLOV (Kotikov, Lipatov, Onishschenko, Velizhanin, hep-th/0404092) noticed (at 2 loops) a remarkable relation between kernels for
  - BFKL evolution (strong rapidity ordering)
  - DGLAP evolution (pdf evolution = strong collinear ordering)
    → **includes cusp anomalous dimension**

  in QCD and N=4 SYM:
- Set fermionic color factor $C_F = C_A$ in the QCD result and keep only the "leading transcendentality" terms. They coincide with the full N=4 SYM result (even though theories differ by scalars)
- Conversely, N=4 SYM results predict pieces of the QCD result

- transcendentality (weight): n for π^n
  n for ζ_n

Similar counting for HPLs and for related harmonic sums used to describe DGLAP kernels at finite j

Sudakov form factor

- Factorization → differential equation for form factor

Mueller (1979); Collins (1980); Sen (1981); Korchemsky, Radyushkin (1987); Korchemsky (1989); Magnea, Sterman (1990)

$$\frac{\partial}{\partial \ln Q^2}\mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$$
$$= \frac{1}{2}\Big[K(\epsilon, \alpha_s) + G(Q^2/\mu^2, \alpha_s(\mu), \epsilon)\Big] \times \mathcal{M}^{[gg\to 1]}(Q^2/\mu^2, \alpha_s(\mu), \epsilon)$$

K, G also obey differential equations (ren. group):

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\Big)(K+G) = 0 \qquad \Big(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\Big)K = -\gamma_K(\alpha_s)$$

cusp anomalous dimension

General amplitude in planar N=4 SYM

- Solve differential equations for K, G. **Easy** because coupling doesn't run.
- Insert result for Sudakov form factor into *n*-point amplitude

$$\Rightarrow \quad \mathcal{M}_n = 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)} = \exp\Big[-\frac{1}{8}\sum_{l=1}^{\infty} a^l \Big(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon}\Big) \sum_{i=1}^{n} \Big(\frac{\mu^2}{-s_{i,i+1}}\Big)^{l\epsilon}\Big] \times h_n$$

loop expansion parameter:

$$a \equiv \frac{N_c \alpha_s}{2\pi}(4\pi e^{-\gamma})^{\epsilon} = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}$$

looks like the one-loop amplitude, but with ε shifted to ($l\ \varepsilon$), up to finite terms

$\hat{\gamma}_K^{(l)}, \hat{\mathcal{G}}_0^{(l)}$ are l-loop coefficients of $\gamma_K(a)$, $\mathcal{G}_0(a) = G(-1, a, \epsilon = 0)$

Rewrite as

$$\mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon, s_{i,i+1})\Big)\Big]$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon\, f_1^{(l)} + \epsilon^2\, f_2^{(l)}$$

collects **3 series of constants**:

$$f_0^{(l)} = \frac{1}{4}\hat{\gamma}_K^{(l)} \qquad f_1^{(l)} = \frac{l}{2}\hat{\mathcal{G}}_0^{(l)} \qquad f_2^{(l)} = (???)$$