Classical Models of Quantum Localization

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Goals:

To describe some simple and special models of **classical dynamics** on the lattice with disorder that exhibit **Anderson localization.**

A) In **2 dimensions** - Localization

Large excursions of the particle from its starting point are rare.

B) In **3 dimensions**: Localization - Diffusion *transition* , as we vary the strength of disorder of the system.

Anderson transition

Classical Reflections of Quantum Dynamics

Examples of Discrete Space Classical Models:

Lorentz Gas models - on $\mathbb{Z}^2$:

Mirror Model
Manhattan Model (Class C quantum network model)

Edge Reinforced random walk - History dependent walk prefers to visit edges it has visited more frequently in the past.

SUSY hyperbolic sigma model on $\mathbb{Z}^3$
Random walk in correlated Random Environment.
Phase transition (Disertori, Sp., Zirnbauer)

Two Hard Problems in Classical Dynamics

A) **Periodic potential** V(x) and $x(t) \in \mathbb{R}^2$

$$\ddot{x}(t) = -\nabla V(x(t))$$

Long time dynamics *poorly* understood.

However, the long time dynamics of quantum Hamiltonian:

$$H = -\Delta + V(x)$$

is ballistic using Bloch wave analysis.

B) **Chirikov's Standard Map**, discrete time pendulum:

$$x_{j+1} + x_{j-1} - 2x_j = K\, sin\, x_j$$

Regular, quasi-periodic dynamics mathematically understood - KAM.

Conjecture: Positive metric entropy - Chaotic motion

Momentum diffusion expected for large K.

Corresponding **quantum** system

$$U = e^{i\alpha\, d^2/dx^2} e^{iK\, cosx}$$

Localization in momentum space.

Casati, Chirikov, Israelev, Ford,
Fishman, Grempel, Prange, Bourgain.

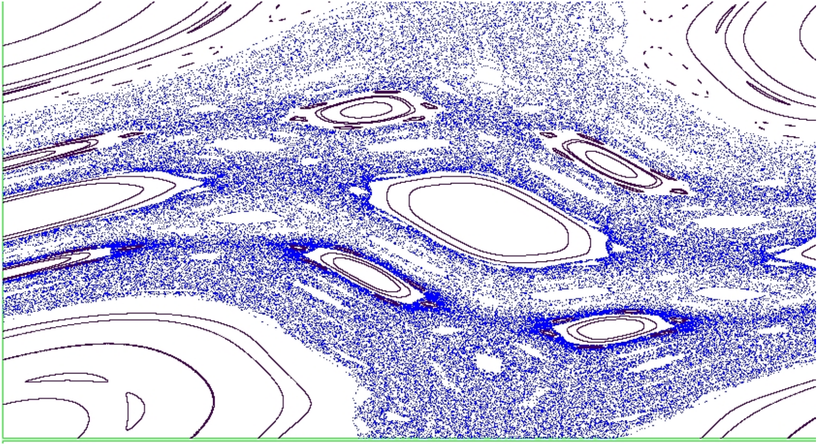

Figure: Orbits of Standard Map

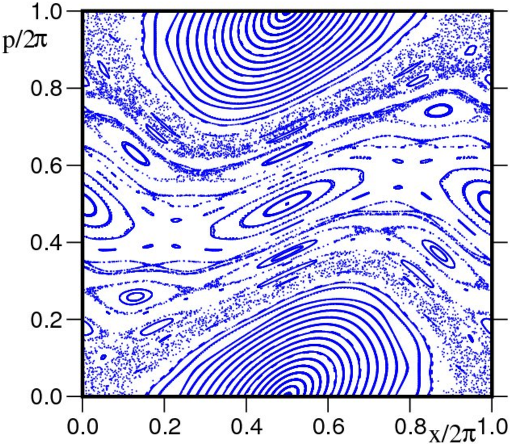

Figure: Oribits of Standard Map

Anderson Localization

Discrete Schrödinger + random potential, $v(j)$, $j \in \mathbb{Z}^d$:

$$i\,\partial\psi/\partial t = H\,\psi = -\Delta\psi + v\psi, \quad -W/2 \le v(j) \le W/2$$

Let $\psi(j,0) = \delta_{j,0}$ and define

$$R^2(t) \equiv \sum |\psi(j,t)|^2 |j|^2.$$

$R^2(t)$ measures the spread of the wave function.

Localization: $\mathbb{E}\, R^2(t) \le Const$

Theorem: Localization holds for large disorder $(W \gg 1)$ for all $d \ge 1$. For d=1, localization for all $W > 0$.

Mathematical conjectures

Conjecture A. In **2D** Localization holds for all $W > 0$

Conjecture B. In **3D** for small $W > 0$, *diffusion* occurs:

$$R^2(t) \approx Dt$$

Thus *Anderson transition* should occur in 3D.

Localization $\rightarrow$ Diffusion

Note that if $v = v(j, t)$ is random in *space and time* then the motion is **always diffusive**, $d \geq 1$

Memory plays a key role in localization.

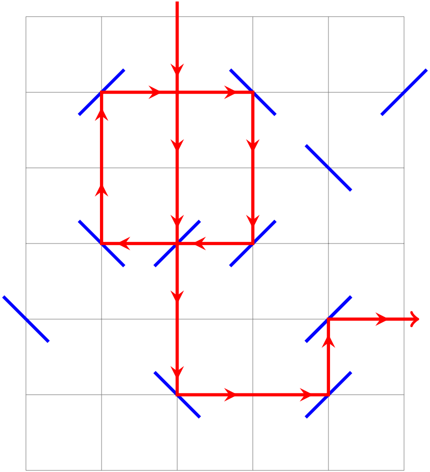

Figure: Cohen's Mirror Model, Mirrors $\pm 45^{\circ}$

Cohen's Mirror Model

Place mirrors at the vertices of a square lattice with concentration $0 \leq C \leq 1$. The mirrors are ± 45 degrees with prob C/2.

No mirror with probability $1 - C$.

If C$=$ 1, the the mirror model $\approx$ **critical percolation.**
All paths are *closed* with probability one.
They are *boundaries of percolation clusters*

If a loop has diameter L, then the length of the loop $\approx L^{7/4}$.

The $<$ *loop length* $>=$ **infinite**!

No localization in 2D

Numerical results for $C < 1$

If $C < 1$, then motion:

$< (x(0) - x(t))^2 >$ looks diffusive or super-diffusive.

Numerics: For $C < 1$, no longer related to percolation on *square* lattice.

Exception: $C < 1$, *triangular* lattice appears to be like critical percolation! Orbits look like boundaries of percolation clusters at criticality.

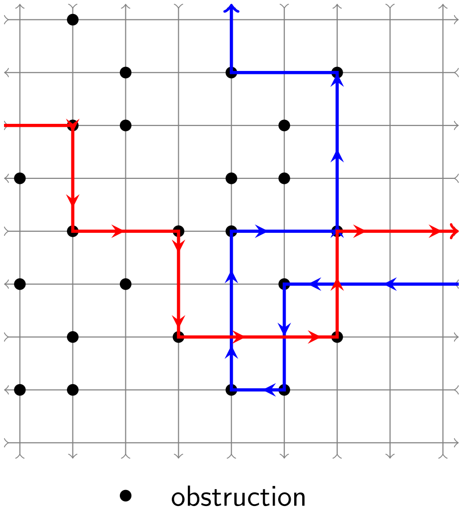

Figure: Manhattan Lattice

Manhattan Model

Due to: Beamond, Cardy, Owczarek; Gruzberg, Ludwig, Read.

Scatterers occur at vertices with probability p.

Equivalent to a quantum network model (Class C)

Theorem. If $p > 1/2$, Then all orbits are closed and

$$< \textit{loop length} > < \infty$$

Conjecture (Beamond et al): For all $p > 0$

$$< \textit{loop length} > \leq \exp\left(C\, p^{-2}\right)$$

Remark: Simulations breakdown for $p \leq .3$ - paths too long.

Edge reinforced Random Walk - ERRW

History dependent walk (Diaconis) :

Walk moves on $\mathbb{Z}^d$, nearest neighbor steps at discrete times t.

Let **n(e, t)** denote the number of times the walk has visited the **edge** e up to time t.

Then the probability $P(v, v', t+1)$ that the walk at vertex v will visit a neighboring edge $e = (v, v')$ at time $t+1$ is

$$P(v, v', t+1) = (\beta + \mathbf{n(e,t)})/S_\beta(v, t)$$

where $S_\beta = \sum(\beta + \mathbf{n(e',t)})$ over all the edges e' touching v.

ERRW as RW in a Random Environment

The generator **D** of the random walk in random environment:

$$\sum_{j\sim j'} |f(j) - f(j')|^2 \, c_{j,j'} \equiv [f, \mathbf{D}(c) f]$$

where $c_{j,j'}$ is the conductance across the edge (j, j').

The **distribution of the conductances** is given by statistical mechanics :

$$Z^{-1} \prod_j r_j^{-1/2} \prod_{j\sim j'} [c_{jj'}^2 / r_j r_{j'}]^\beta \sqrt{\det \mathbf{D}(c)} \prod_{j\sim j'} \frac{dc_{j,j'}}{c_{jj'}}$$

where $r_j \equiv \sum_{i\sim j} c_{ji}$. **Pin at 0**: $c_{0,j} = 1$

Results and Conjectures for ERRW

Theorem (Merkl-Rolles) In **1D** ERRW is **localized**,

$$\mathbb{E}c_{j,j'}^{1/4} \leq e^{-m|j|} \;\rightarrow\; Prob[W(t)| \geq |j|] \leq e^{-|j|/\beta}$$

In **2D**, $\mathbb{E}c_{j,j'}^{1/4} \rightarrow 0$ for large $|j|$. (Mermin-Wagner).

Theorem: There is a phase transition on the Bethe Lattice. Transient for β large to recurrent, β small.

Conjecture: In **3D** there is a phase transition: Localization, for small β, to diffusion, for large β.

Anderson Transition for a 3D SUSY hyperbolic Sigma Model (Disertori, Sp., Zirnbauer)

In **3D** a *simplified* version of Efetov's SUSY lattice field theory has an *Anderson transition*: **Localization** $\rightarrow$ **Diffusion.**

The femions can be integrated out producing and a real Effective Action $A_\beta(t)$, where t_j are real variables.

Correlations can be expressed as a **random walk** in a **random environment** with random conductances:

$$\beta e^{t_j + t_{j'}}, \text{ across each edge } j \sim j'.$$

Effective Action

The **distribution of the conductances** is similar to ERRW:

$$e^{-A_\beta(t)} = \prod_{j \sim j'} e^{-\beta cosh(t_j - t_{j'})} \sqrt{Det\ D(c)} \prod_j e^{-t_j} dt_j$$

where $t_j \in \mathbb{R}$ and $c_{jj'} = e^{t_j + t_{j'}}$. **Pin** $t_0 = 0$.

Theorem.(DSZ) In 3D, if β is large the conductance, $e^{t_j + t_{j'}} \approx 1$ for all edges (j, j') with high probability.
Thus motion is **"diffusive"**.

Theorem.(DS) If β is small then for all dimensions, the conductance $\to 0$,

$$< e^{(t_j + t_{j'})/4} > \leq e^{-m|j|} \quad m > 0.$$

Thus motion is exponentially **localized**.

Conclusions and Speculations

Some discrete lattice classical dynamics have the analog of Anderson Localization and Delocalization.

The Hyperbolic SUSY model and ERRW walk have many striking similarities.

The Manhattan models are perhaps related to a Heisenberg SUSY model. They have some subtle self attraction properites.

ERRW and Hyperbolic SUSY model are toy models for understanding *new universality classes*.
Do they have an upper critical dimension?