A Solution to the Helmholtz Equations in Oblate Spheroidal Coordinates for Tapered Fiber Mode Analysis

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Abstract

In this project, my initial goal was to find an analytical solution to the problem of analyzing a tapered fiber's confined modes, for the purpose of implementation in research of the optical properties of Muller cells found in the human eye.

Following research conducted in Dr. Ribak's lab and in many others [1] on Muller cells, cells found inside the human eye inside the transparent tissue directly above the cone photo-receptors, which can be thought of as weakly guiding tapered optical fibers. Apparently, these cells permit the propagation of light with specific wavelengths inside them and onto the cone cells in the retina, while dispersing the rest of the incident light onto the rod cells, surrounding the cones cells.

For the purpose of this research, an analytical method for the calculation of approximate modes propagating inside tapered fibers, could complement numerical work done by Dr. Ribak and Amichai Labin [2] and perhaps provide more insight about the nature of light propagation inside a tapered medium.

While I initially planned on approaching the problem using the conventional tool used for tapered fibers, the Coupled Mode Theory (which from now on will be referred to as CMT) [3][4], I have ended up using a novice and different, more "intuitive" method. In this method, making some use of the formalism previously presented by [5] for Gaussian Beams, I do calculations in a more "natural", orthonormal coordinate system, called the oblate spheroidal coordinates.
The motivation to do so, lies in the fact that in this system, the fiber's boundary conditions are constant.
In such a system, since Helmholtz Equation is separable [6], there is no coupling between the different coordinates, thus eliminating the need for CMT.

The Oblate Spheroidal Coordinate System

In this report, I will describe the way to received an approximate solution to the Helmholtz Equation in a set of orthonormal coordinates, called the Oblate Spheroidal Coordinates, using the paraxial approximation. These curvilinear coordinates, are constructed of hyperbolas that intersect with ellipses at normal angles, thus defining a set of orthogonal coordinates [7]. Figure 1 depicts these coordinates projected onto the Z-X Cartesian plane

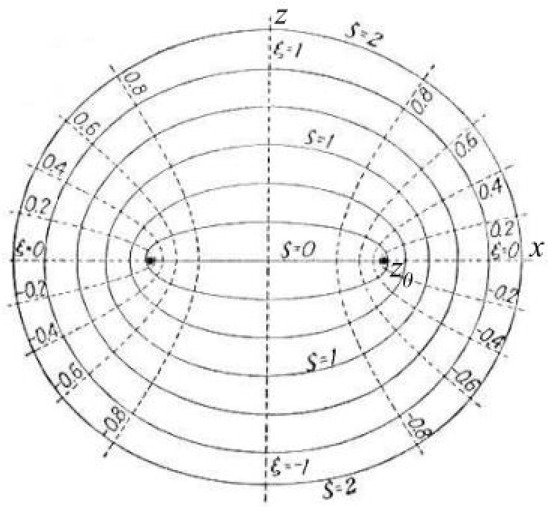

Figure 1 The Oblate Spheroidal Coordinates [5]

In these coordinates ζ , representing the ellipses, takes on values between $0 < \zeta < \infty$ while ξ represent the hyperbolas and takes on values $-1 < \xi < 1$.

Notice that both ξ and ζ and dimensionless.

z_0 is defined as the distance between the origin and the foci of both ellipses and hyperbolas, whose surfaces of revolution about the z axis are single sheeted hyperbolas of

constant ξ and ζ .

These coordinates obey the following transformation to the three Cartesian Coordinates and the radial Cylindrical Coordinate, which is presented for later use

$$\begin{cases} x = z_0\sqrt{1+\zeta^2}\sqrt{1-\xi^2}cos(\phi) \\ y = z_0\sqrt{1+\zeta^2}\sqrt{1-\xi^2}sin(\phi) \\ z = z_0\zeta\xi \\ r = \sqrt{x^2+y^2} = z_0\sqrt{(1+\zeta^2)(1-\xi^2)} \end{cases} \quad (1)$$

With the following scale factors

$$\begin{cases} h_\zeta = z_0\sqrt{\frac{\zeta^2+\xi^2}{1+\zeta^2}} \\ h_\xi = z_0\sqrt{\frac{\zeta^2+\xi^2}{1-\xi^2}} \\ h_\phi = z_0\sqrt{(1+\zeta^2)(1-\xi^2)} \end{cases} \quad (2)$$

Separation of Variables

What I now show is the solution for the E_ζ component of the electric field, using the Helmholtz equation (A scalar wave equation, which assumes the same temporal dependence for all field components) [8].
In the solution of E_ζ one should keep in mind I eventually want to solve propagation along a waveguide. This means I assume the same ζ dependence as well as the previously mentioned temporal dependence for all of the electric and magnetic field components [9]. This translates as the requirement $\begin{bmatrix} E(\vec{\mathbf{r}},t) \\ H(\vec{\mathbf{r}},t) \end{bmatrix} = \begin{bmatrix} E(\xi,\phi) \\ H(\xi,\phi) \end{bmatrix} Z(\zeta)e^{i\omega t}$.

In this coordinate system, the Helmholtz equation

$$\nabla^2 E_\zeta + k^2 E_\zeta = 0 \quad (3)$$

can be written as

$$\frac{\partial}{\partial\zeta}(1+\zeta^2)\frac{\partial E_\zeta}{\partial\zeta} + \frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial E_\zeta}{\partial\xi} + \frac{\xi^2+\zeta^2}{(1+\zeta^2)(1-\xi^2)}\frac{\partial^2 E_\zeta}{\partial\phi^2} + z_0^2k^2(\zeta^2+\xi^2)E_\zeta = 0 \quad (4)$$

Where the Laplacian is

$$\nabla^2 E_\zeta = \frac{1}{z_0^2(\zeta^2+\xi^2)}\left[\frac{\partial}{\partial\zeta}(1+\zeta^2)\frac{\partial E_\zeta}{\partial\zeta} + \frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial E_\zeta}{\partial\xi} + \frac{\xi^2+\zeta^2}{(1+\zeta^2)(1-\xi^2)}\frac{\partial^2 E_\zeta}{\partial\phi^2}\right] \quad .(5)$$

I redefine ξ in the following manner

$$\sigma \equiv \frac{b}{2}(1-\xi^2) \quad (6)$$

$$\downarrow$$

$$d\sigma = \frac{b}{2}(-2\xi d\xi) \quad (7)$$

where I define b as follows

$$b \equiv \frac{k^2 z_0^2}{k_\xi z_0} = k^2\frac{z_0}{k_\xi} \quad (8)$$

$k = \frac{2\pi}{\lambda}$ is the wave number and k_ν is the wave number in the direction ν.

Under the assumption of paraxial propagation [10], I assume propagation to take place mostly inside a narrow, single sheeted hyperbola defined by a constant ξ where

$$\xi \sim 1\,. \quad (9)$$

This assumption, defines **guided** and **positive** z axis propagation. I can substitute (9) into (7) and receive

$$d\sigma \approx -bd\xi \rightarrow \frac{d}{d\xi} \approx -b\frac{d}{d\sigma} \quad .(10)$$

Substituting (6) and (10) into (4) I get the following equation

$$\frac{\partial}{\partial\zeta}\left[(1+\zeta^2)\frac{\partial E_\zeta}{\partial\zeta}\right]+2b\frac{\partial}{\partial\sigma}\left[\sigma\frac{\partial E_\zeta}{\partial\sigma}\right]+\left(\frac{1-\frac{2\sigma}{b}+\zeta^2}{(1+\zeta^2)\frac{2\sigma}{b}}\right)\frac{\partial^2 E_\zeta}{\partial\phi^2}+z_0^2k^2(\zeta^2+1-\frac{2\sigma}{b})E_\zeta=0 \quad .(11)$$

I can now try and solve the Helmholtz equation, under the above assumptions and substitutions, for E_ζ.

Since the oblate spheroidal system maintains azimuthal symmetry, the ϕ dependence of E_ζ is trivial and because I am are looking for a solution which will propagate on the ζ axis, I seek a separable solution for E_ζ of the form

$$E_\zeta(\sigma,\zeta,\phi)=Z_m(\zeta)S_m(\sigma)e^{\pm im\phi} \quad (12)$$

where $m \epsilon Integers$. Substituting (12) into (11) I arrive at

$$S_m(\sigma)\frac{\partial}{\partial\zeta}(1+\zeta^2)\frac{\partial}{\partial\zeta}Z_m(\zeta)+2Z_m(\zeta)\frac{\partial}{\partial\sigma}\sigma\frac{\partial}{\partial\sigma}S_m(\sigma)-m^2Z_m(\zeta)S_m(\sigma)$$
$$\left(\frac{1}{2\sigma}-\frac{1}{(1+\zeta^2)}\right)+z_0^2k^2(1+\zeta^2-2\sigma)Z_m(\zeta)S_m(\sigma)=0 \quad .(13)$$

This equation is separable and separates into

$$\begin{cases}\frac{\partial}{\partial\zeta}(1+\zeta^2)\frac{\partial}{\partial\zeta}Z_m(\zeta)+Z_m(\zeta)\left(\frac{m^2}{(1+\zeta^2)}+z_0^2k^2+z_0^2k^2\zeta^2+\lambda_m^2\right)=0 & (14)\\ 2b\frac{\partial}{\partial\sigma}\sigma\frac{\partial}{\partial\sigma}S_m(\sigma)-S_m(\sigma)\left(\frac{2\sigma z_0^2k^2}{b}+m^2\frac{b}{2\sigma}+\lambda_m^2\right)=0 & (15)\end{cases}$$

where λ_m^2 is the separation constant. Since the equation for ζ is unbounded, λ_m^2 plays the role of k_ζ^2.

Solving the E_ζ Field Components

I will start solving (14). This equation is a second order, ordinary differential equation. Since I am interested in eventually describing all field components as oscillating functions of ζ, I try to solve the equation for ζ using a plane wave with a complex envelope

$$Z_m(\zeta) = A_m(\zeta) e^{ik_\zeta z_0 \zeta} \quad .(16)$$

Under the paraxial approximation, for nearly axial solutions, I can require $A_m(\zeta)$ to be a slowly varying envelope. This requirement, used along with (16) and neglecting terms smaller than k or k_ζ (which are approximately 10^7 for visible light), takes us from

$$(1+\zeta^2)\frac{\partial^2 A_m(\zeta)}{\partial \zeta^2} + 2\left(\zeta + ik_\zeta + i\zeta^2 k_\zeta\right)\frac{\partial A_m(\zeta)}{\partial \zeta} +$$
$$+A_m(\zeta)\left(\frac{m^2}{(1+\zeta^2)} + z_0^2 k^2 + \zeta^2\left(z_0^2 k^2 - z_0^2 k_\zeta^2\right) + 2i\zeta k_\zeta\right) = 0 \quad (17)$$

to

$$\frac{\partial A_m(\zeta)}{\partial \zeta} + A_m(\zeta)\frac{\left(z_0^2 k^2 + \zeta^2\left(z_0^2 k^2 - z_0^2 k_\zeta^2\right) + 2\zeta i k_\zeta\right)}{2k_\zeta i(1+\zeta^2)} = 0 \quad .(18)$$

The solution to equation (18) is

$$A_m(\zeta) = exp\left(\frac{i\left(Arctan(\zeta)\left(k^2\zeta^2 - k_\xi^2\right) + k_\xi^2 \zeta\right) - k_\zeta log(\zeta^2+1)}{2k_\zeta}\right) \quad (19)$$

and so the solution for $Z_m(\zeta)$ is

$$Z_m(\zeta) = \frac{1}{\sqrt{(\zeta^2+1)}} exp\left(\frac{i\left(Arctan(\zeta)\left(k^2\zeta^2 - k_\xi^2\right) + \zeta\left(k_\xi^2 + k_\zeta z_0\right)\right)}{2k_\zeta}\right) \quad .(20)$$

I now turn to solve the equation (15) for σ. This equation is also a second order, ordinary differential equation. Substituting (8) into (15) I get

$$\frac{\partial}{\partial \sigma}\sigma\frac{\partial}{\partial \sigma}S_m(\sigma) - S_m(\sigma)\left(\frac{k_\xi^2}{k^2}\sigma + \frac{m^2}{4\sigma} - \frac{k_\xi z_0}{2}\frac{k_\zeta^2}{k^2}\right) = 0 \quad .(21)$$

In order to bring (21) to the form of a known differential equation, I use the following substitutions. In order to get rid of the term $-S_m(\sigma)\frac{m^2}{4\sigma}$ in (21), the following substitution is used

$$S_m(\sigma) = \sigma^{\pm\frac{m}{2}} R_m(\sigma) \quad (22)$$
$$\downarrow$$
$$\sigma\frac{\partial^2 R_m(\sigma)}{\partial \sigma^2} + \frac{\partial R_m(\sigma)}{\partial \sigma}(1 \pm m) - R_m(\sigma)\left(\frac{k_\xi^2}{k^2}\sigma - \frac{k_\xi z_0}{2}\frac{k_\zeta^2}{k^2}\right) = 0 \quad (23)$$

where the $\pm$ sign is chosen so that $\pm\frac{m}{2} \geq 0$. In order to get rid of the term $-R_m(\sigma)\frac{k_\xi^2}{k^2}\sigma$ I use

$$R_m(\sigma) = e^{-\frac{k_\xi^2}{k^2}\sigma} F_m(\sigma) \tag{24}$$

$$\downarrow$$

$$\sigma\frac{\partial^2 F_m(\sigma)}{\partial\sigma^2} + \frac{\partial F_m(\sigma)}{\partial\sigma}\left(1 \pm m - 2\frac{k_\xi^2}{k^2}\sigma\right) + F_m(\sigma)\left(\frac{k_\zeta^2 k_\xi z_0 - 2k_\xi^2(1\pm m)}{2k^2}\right) = 0 \tag{25}$$

Equation (25) is in the form of the confluent hypergeometric differential equation [91], for which the general solutions depend on the sign of the coefficient $\frac{k_\xi^2}{k^2}$ as

$$k^2 = k_\xi^2 + k_\zeta^2 \quad (26)$$

$$\downarrow$$

$$\frac{k_\xi^2}{k^2} = 1 - \frac{k_\zeta^2}{k^2} \quad .(27)$$

This separates the general solution to (25) into two regions

$$\begin{cases} F_m(\sigma) = C_1 U\left(-\frac{4k_\xi}{k_\zeta^2 z_0 - 2k_\xi(1\pm m)}, 1 \pm m, 2\frac{k_\xi^2}{k^2}\sigma\right) + \\ +C_2 L\left(\frac{4k_\xi}{k_\zeta^2 z_0 - 2k_\xi(1\pm m)}, \pm m, 2\frac{k_\xi^2}{k^2}\sigma\right) & 1 \gg \frac{k_\xi^2}{k^2} \geq 0 \\ F_m(\sigma) = e^{-2\frac{k_\xi^2}{k^2}\sigma} C_3 U\left(\frac{1\pm m}{2} + \frac{k_\xi z_0}{4}\frac{k_\zeta^2}{k_\xi^2}, 1 \pm m, 2\frac{k_\xi^2}{k^2}\sigma\right) + \\ +e^{-2\frac{k_\xi^2}{k^2}\sigma} C_4 L\left(\frac{1\pm m}{2} + \frac{k_\xi z_0}{4}\frac{k_\zeta^2}{k_\xi^2}, \pm m, 2\frac{k_\xi^2}{k^2}\sigma\right) & \frac{k_\xi^2}{k^2} < 0 \end{cases} \tag{28}$$

where inside the fiber's core $1 \gg \frac{k_\xi^2}{k^2} \geq 0$ and in the fiber's
cladding $\frac{k_\xi^2}{k^2} < 0$. U is the confluent hypergeometric function of the second kind and L are generalized Laguerre polynomials. Inspection of the asymptotic form, for small and large values of $2\frac{k_\xi^2}{k^2}\sigma$, for both the confluent hypergeometric function of the second kind and the Laguerre polynomials, will reveal the private solution to (25). Requiring non-divergent solutions, will determine which function will constitute a solution in the different regions. For a finite value of$m$, the function $U\left(\frac{1\pm m}{2} + \frac{k_\xi z_0}{4}\frac{k_\zeta^2}{k_\xi^2}, 1 \pm m, 2\frac{k_\xi^2}{k^2}\sigma\right)$ diverges as $2\frac{k_\xi^2}{k^2}\sigma \longrightarrow 0$ and is final at $2\sigma \longrightarrow \infty$. The opposite is true for $L\left(\frac{1\pm m}{2} + \frac{k_\xi z_0}{4}\frac{k_\zeta^2}{k_\xi^2}, \pm m, 2\frac{k_\xi^2}{k^2}\sigma\right)$.
For these reasons the solution for (25), taking the cladding to infinity as usually done for fibers [12], is

$$\begin{cases} F_m(\sigma) = C_2 L\left(\frac{4k_\xi}{k_\zeta^2 z_0 - 2k_\xi(1\pm m)}, \pm m, 2\frac{k_\xi^2}{k^2}\sigma\right) & \textit{core} \\ F_m(\sigma) = e^{-2\frac{k_\xi^2}{k^2}\sigma} C_3 U\left(\frac{1\pm m}{2} + \frac{k_\xi z_0}{4}\frac{k_\zeta^2}{k_\xi^2}, 1 \pm m, 2\frac{k_\xi^2}{k^2}\sigma\right) & \textit{cladding} \end{cases} \tag{29}$$

and so the solution for $S_m(\sigma)$ is

$$\begin{cases} S_m(\sigma) = C_2 e^{-\frac{k_\xi^2}{k^2}\sigma} \sigma^{\pm\frac{m}{2}} L\left(\frac{4k_\xi}{k_\zeta^2 z_0 - 2k_\xi(1\pm m)}, \pm m, 2\frac{k_\xi^2}{k^2}\sigma\right) & core \\ S_m(\sigma) = C_3 e^{-\frac{k_\xi^2}{k^2}\sigma} \sigma^{\pm\frac{m}{2}} U\left(\frac{1\pm m}{2} + \frac{k_\xi z_0}{4}\frac{k_\zeta^2}{k_\xi^2}, 1\pm m, 2\frac{k_\xi^2}{k^2}\sigma\right) & cladding \end{cases} .(30)$$

This can be written for $S_m(\xi)$ as follows

$$\begin{cases} S_m(\xi) = C_2 e^{-\frac{1}{2}k_\xi z_0(1-\xi^2)} \left(\frac{k^2 z_0(1-\xi^2)}{2k_\xi}\right)^{\pm\frac{m}{2}} L\left(\frac{4k_\xi}{k_\zeta^2 z_0 - 2k_\xi(1\pm m)}, \pm m, k_\xi z_0(1-\xi^2)\right) & \xi_0 < |\xi| < 1 \\ S_m(\xi) = C_3 e^{-\frac{1}{2}k_\xi z_0(1-\xi^2)} \left(\frac{k^2 z_0(1-\xi^2)}{2k_\xi}\right)^{\pm\frac{m}{2}} U\left(\frac{1\pm m}{2} + \frac{k_\xi z_0}{4}\frac{k_\zeta^2}{k_\xi^2}, 1\pm m, k_\xi z_0(1-\xi^2)\right) & 0 < |\xi| < \xi_0 \end{cases} \quad (31)$$

for ξ_0 being the core-cladding interface.

And so, finally, the solution to the Helmholtz Equation for E_ζ is approximately given by

$$E_\zeta(\xi,\zeta,\phi) = Z_m(\zeta) S_m(\xi) e^{\pm im\phi} \approx$$
$$\frac{S_m(\xi)}{\sqrt{(\zeta^2+1)}} exp\left(\frac{i\left(Arctan(\zeta)\left(k^2\zeta^2 - k_\xi^2\right) + \zeta\left(k_\xi^2 + k_\zeta z_0\right) \pm m\phi\right)}{2k_\zeta}\right), \quad (32)$$

The same procedure can also be done for the magnetic field component H_ζ which will bear the same solutions.

Fiber Boundary Conditions

Now once I have found the field components E_ζ and H_ζ, I can move on to calculating the other non radial field components E_ϕ, H_ϕ which will be necessary in order satisfy the boundary conditions.
As a reminder, the reason to begin this algebric exercise in the first place was to have **constant** boundary conditions, instead of the z dependent ones we would have gotten using the polar coordinate system. In order to receive the rest of the field components, one must use Maxwell's equations (more specifically, Ampere's and Faraday's laws). This is no trivial ordeal since the curl in said equations, if carried out using "brute force", will couple the ζ coordinate along with the rest of the coordinates. Doing so will render the above method completely useless as it will produce ζ coupled solutions, that will require the use of numerical calculations to solve. I do have an idea of how this can be done but, after consulting with my supervisor, we have decided the complete solution is outside the scope of the academic project course. I will elaborate on the method in which I believe the rest of the field components can be calculated but will not provide a complete calculation.

The fibers boundary conditions require continuity of $E_\zeta, E_\phi H_\zeta, H_\phi$ at the core cladding interface (denoted with the superscript cr and cl respectively), where the refraction index non-continuously changes from n_1 to n_2. I write the E_ζ and H_ζ field components inside and outside the fiber's core (where the interface is at $\xi = \xi_0$)

$$\begin{cases} E_\zeta^{cr}(\zeta,\xi,\phi) = H_\zeta^{cr}(\zeta,\xi,\phi) = C_2 \frac{\left(\frac{k^2 z_0(1-\xi^2)}{2k_\xi}\right)^{\pm\frac{m}{2}} L\left(\frac{4k_\xi}{k_\zeta^2 z_0 - 2k_\xi(1\pm m)}, \pm m, k_\xi z_0(1-\xi^2)\right)}{\sqrt{(\zeta^2+1)}} \\ exp\left(\frac{i\left(Arctan(\zeta)\left(k^2\zeta^2 - k_\xi^2\right) + \zeta\left(k_\xi^2 + k_\zeta z_0\right) \pm m\phi\right) - \frac{1}{2}k_\xi z_0(1-\xi^2)}{2k_\zeta}\right) & \xi_0 < |\xi| < 1 \\ E_\zeta^{cl}(\zeta,\xi,\phi) = H_\zeta^{cl}(\zeta,\xi,\phi) = \frac{C_3\left(\frac{k^2 z_0(1-\xi^2)}{2k_\xi}\right)^{\pm\frac{m}{2}} U\left(\frac{1\pm m}{2} + \frac{k_\xi z_0}{4}\frac{k_\zeta^2}{k_\xi^2}, 1\pm m, k_\xi z_0(1-\xi^2)\right)}{\sqrt{(\zeta^2+1)}} \\ exp\left(\frac{i\left(Arctan(\zeta)\left(k^2\zeta^2 - k_\xi^2\right) + \zeta\left(k_\xi^2 + k_\zeta z_0\right) \pm m\phi\right) - \frac{1}{2}k_\xi z_0(1-\xi^2)}{2k_\zeta}\right) & 0 < |\xi| < \xi_0 \end{cases} \quad (33)$$

From Maxwell's equation I can calculate the rest of the E and H components as is usually done for cylindrical coordinates [11].
The "brute force" method includes making use of the scale factors (2) to calculate the curls needed for Maxwell's Equations

$$\begin{cases} \nabla \times H = \frac{\partial E}{\partial t} = -i\omega E & (34) \\ \nabla \times E = -\frac{\partial H}{\partial t} = i\omega H & (35) \end{cases} .$$

The curl in an orthonormal coordinate system, can generally be written as

$$(\nabla \times F)_k = \frac{h_k \hat{e}_k}{H} \epsilon_{ijk} \frac{\partial}{\partial q^i}(h_j F_j) \quad (36)$$

where I define $H \equiv h_i h_j h_k$ and h_{ijk} are the coordinate scale factors. Since I'm considering scalar field components, where polarization is irrelevant, I can write the curl for the oblate spheroidal coordinates as follows

$$-i\omega E_\zeta = \frac{1}{h_\xi h_\phi}[\frac{\partial}{\partial \phi}(h_\xi H_\xi) - \frac{\partial}{\partial \xi}(h_\phi H_\phi)] \quad (37)$$
$$-i\omega E_\xi = \frac{1}{h_\zeta h_\phi}[\frac{\partial}{\partial \zeta}(h_\phi H_\phi) - \frac{\partial}{\partial \phi}(h_\zeta H_\zeta)] \quad (38)$$
$$-i\omega E_\phi = \frac{1}{h_\zeta h_\xi}[\frac{\partial}{\partial \xi}(h_\zeta H_\zeta) - \frac{\partial}{\partial \zeta}(h_\xi H_\xi)] \quad (39)$$
$$i\omega H_\zeta = \frac{1}{h_\xi h_\phi}[\frac{\partial}{\partial \phi}(h_\xi E_\xi) - \frac{\partial}{\partial \xi}(h_\phi E_\phi)] \quad (40)$$
$$i\omega H_\xi = \frac{1}{h_\zeta h_\phi}[\frac{\partial}{\partial \zeta}(h_\phi E_\phi) - \frac{\partial}{\partial \phi}(h_\zeta E_\zeta)] \quad (41)$$
$$i\omega H_\phi = \frac{1}{h_\zeta h_\xi}[\frac{\partial}{\partial \xi}(h_\zeta E_\zeta) - \frac{\partial}{\partial \zeta}(h_\xi E_\xi)] \quad .(42)$$

Now I can calculate E_ϕ and H_ϕ using our known solutions for H_ζ E_ζ

$$-i\omega E_\phi = \frac{1}{h_\zeta h_\xi}\left[\frac{\partial}{\partial \xi}(h_\zeta H_\zeta) + \frac{\partial}{\partial \zeta}\left\{\frac{ih_\xi}{\omega h_\zeta h_\phi}[\frac{\partial}{\partial \zeta}(h_\phi E_\phi) - \frac{\partial}{\partial \phi}(h_\zeta E_\zeta)]\right\}\right]. \quad (43)$$

In order to receive analytical solutions for E_ϕ,H_ϕ, I again would use the paraxial approximation. In this case approximating the scale factors only for the "portion" of space I am interested to find solutions in. The approximate scale factors, approximating$\sigma \sim 0$, are

$$\begin{cases} h_\zeta = z_0\sqrt{\frac{\zeta^2+\xi^2}{1+\zeta^2}} = z_0\sqrt{1-\frac{2\sigma}{b(1+\zeta^2)}} \approx z_0 & (44) \\ h_\sigma = z_0\sqrt{\frac{\zeta^2+\xi^2}{1-\xi^2}} = z_0\sqrt{\frac{b(\zeta^2+1)}{2\sigma}-1} \approx z_0\sqrt{\frac{b(\zeta^2+1)}{2\sigma}} & (45) \\ h_\phi = z_0\sqrt{(1+\zeta^2)(1-\xi^2)} = z_0\sqrt{\frac{2\sigma(1+\zeta^2)}{b}}. & (46) \end{cases}$$

These would be used to calculate the needed field components. The problem is, these expressions are far from elegant and in order to remove the coupling of ζ to the rest of the coordinates, many terms have to be neglected.
A different approach would reason that since the zero order of the oblate spheroidal coordinates are simply the polar coordinates, one can use the common [11] polar curl in order to derive E_ϕ and

$$\begin{cases} E_\phi = \frac{-ik_z}{k^2-k_z^2}\left(\frac{\partial}{r\partial \phi}E_z - \frac{\omega\mu}{k_z}\frac{\partial}{\partial r}H_z\right) & (47) \\ H_\phi = \frac{-ik_z}{k^2-k_z^2}\left(\frac{\partial}{r\partial \phi}E_z + \frac{\omega\mu}{k_z}\frac{\partial}{\partial r}H_z\right) & (48) \end{cases}$$

This can easily be confirmed when one takes the limit $\zeta \longrightarrow 0$ or $\xi \longrightarrow 1$ with

ϕ unaffected. In the first limit

$$r = \sqrt{x^2 + y^2} = z_0\sqrt{(1+\zeta^2)(1-\xi^2)} \quad (49)$$

$$\downarrow$$

$$lim_{\zeta \longrightarrow 0} r = z_0\sqrt{1-\xi^2} \quad .(50)$$

This allows calculation of the approximate derivative of r in terms of the oblate spheroidal coordinates

$$dr \approx -z_0 \frac{\xi}{\sqrt{1-\xi^2}} d\xi \quad (51)$$

$$\frac{\partial}{\partial r} \approx -\frac{\sqrt{1-\xi^2}}{z_0 \xi} \frac{\partial}{\partial \xi} \quad .(52)$$

This approach can be used to calculate, without coupling of the ζ coordinate, the needed field components and give an approximate solution to the problem at hand. Since, as previously mentioned, numerical work has been done on the subject, this method will be followed through and compared with numerical results.

Once the remaining fields are found, boundary conditions can be demanded

$$\begin{cases} E_\phi(n_{cr}) = E_\phi(n_{cl}) \\ H_\phi(n_{cr}) = H_\phi(n_{cl}) \\ E_\zeta(n_{cr}) = E_\zeta(n_{cl}) \\ H_\zeta(n_{cr}) = H_\zeta(n_{cl}). \end{cases} \quad (53)$$

and a non trivial solution can be found, thus defining the fiber modes.

Conclusions

In this project, I have shown analytical solutions to the Helmholtz equation for the field components E_ζ and H_ζ in oblate spheroidal coordinates, under the assumption of paraxial propagation. I have also shown two methods by which the other non-radial field components E_ϕ and H_ϕ may be calculated and paved the way for the complete mode analysis of a tapered fiber, which displays oblate spheroidal symmetry.

I believe that this different approach to tapered fiber mode analysis could prove useful in the analysis of tapers displaying different symmetries, not only the ones offered by the oblate spheroidal coordinates, since the Helmholtz equation is separable in 11 different orthonormal coordinate systems in total [5]. This means that the same process may be done for any fiber who's shape can be approximated to one of these 11 systems and can therefore, be solved analytically for paraxial propagation. Finally, this approach might find use in any other field in which physical phenomena are described as paraxial waves propagating in "tapered" media.

Doing this project, I have learned a great deal about many aspect of theoretical research. I have gone from having no knowledge at all regarding my research subject, to diligently searching for and gathering relevant research materials, learned about standard methods and conventions for solving the canonical problem at hand and all the way through coming up with a new, interesting way to approach the problem I was required to solve. I am confident this new set of tools will greatly assist me in my upcoming studies as well as in future research.

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