Enhancement of surface photo-currents in C3V topological insulators using magnetic super-lattices

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abstract:

In paper [1] it was shown that Photo-currents are theoretically possible in topological insulators in an isotropic Dirac cone model, if one would brake the rotational symmetry with stripes of magnetic substance that is placed on the material. This magnetic stripes can be represented by a periodic magnetic field. In this work it is shown ,with a simulation, that this phenomenon can exist in a more common topological insulator with C3V symmetry instead of a rotational one. more over, this symmetry will inflict new properties to the photo-current such as a light frequency dependent angle, and an increase of power over a range of high frequencies compared with the isotropic case.

1 Describing the model

The Hamiltonian describing the surface electrons is [2]:

$$H_0 = v_f\left[\left(p_x\sigma^y - p_y\sigma^x\right) + c\left(p_+^3 + p_-^3\right)\sigma^z\right]$$

where v_f is Fermi velocity, $p = (p_x, p_y) = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}, \frac{\hbar}{i}\frac{\partial}{\partial y}\right)$ is the momentum operator.$\sigma^x, \sigma^y, \sigma^z$ are Pauli matrices. $p_+ = p_x + ip_y$, $p_- = p_x - ip_y$ and $c = \frac{\lambda}{2\hbar^2}$ is the strength of the rotational symmetry breaking by the C3V expression.

1.1 The symmetries of the unperturbed Hamiltonian

1) time reversal: $TH_0T^{-1} = H_0$ with $T = i\sigma_y K$ for half spin and K is the conjugate operator

2) C_3V symmetry: the Hamiltonian is symmetric in operator C_3 that transforms the state as:

$C_3\ :\ p_\pm \to e^{\pm i2\pi/3}p_\pm,\ \sigma_\pm \to e^{\pm i2\pi/3}\sigma_\pm,\ \sigma_z \to \sigma_z$

3) symmetry under the reflection transformation $M = \Pi_x\sigma_x$ where:

$\Pi_x\, x = -x$, $\Pi_x\, y = y\,\Pi_x\, z = z$ so:

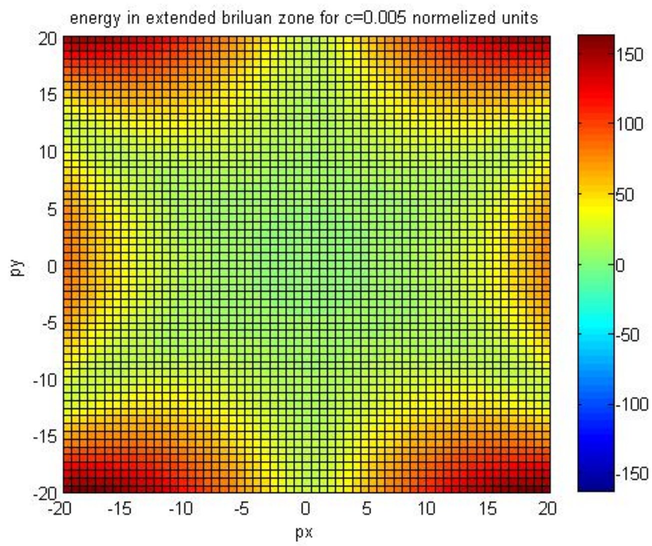
$M:\ k_+ \leftrightarrow -k_-,\ \sigma_x \leftrightarrow \sigma_x,\ \sigma_{y,z} \leftrightarrow -\sigma_{y,z}$

$M:\ k_x = \frac{1}{2}\left(k_+ + k_-\right) \to \frac{1}{2}\left(-k_- - k_+\right) = -k_x, \quad k_y = \frac{1}{2}\left(k_+ - k_-\right) \to \frac{1}{2}\left(-k_- + k_+\right) = k_y$

magnetic stripes effect:

4) reflection transformation: $M_y = \Pi_y \sigma_y$: in this case the operation M_y on the Hamiltonian doesn't change the Eigen energies but do change the Eigen vectors in the following way: $M_y \breve{H}_0 M_y^{-1} = M_y v_f \left(p_x \sigma^y - p_y \sigma^x\right) M_y^{-1} + M_y c \left(p_+^3 + p_-^3\right) \sigma^z M_y^{-1} = v_f \left(p_x \sigma^y - p_y \sigma^x\right) - c \left(p_+^3 + p_-^3\right) \sigma^z$

proof of these symmetries are in appendix 1.



1. Figure 1: energy level of H_0 in extended briluan zone

1.2 Appliance of periodic in x and constant in y magnetic field

For the purpose of symmetry breaking, a periodic magnetic field in y and constant in x on the surface represented by $V = \bar{u} \cdot \bar{\sigma} cos\left(qy\right)$ is added to the system.

where $\bar{u}$ is the magnetic field amplitude, and q is the wave number, the Hamiltonian now is:

$$H = H + V$$

in this periodicity its convenient to use reduced BZ of $k_y \in \left[-\frac{q}{2}, \frac{q}{2}\right]$

the current is written as in [1] with the difference only in the $M_{mn}^{\alpha\beta}$ component:

$$j_k = \frac{e\tau}{\hbar} \left(ev_f\right)^2 \left(\frac{1}{\omega^2}\right) E_m\left(\omega\right) Q_{kmn}\left(\omega\right) E_n * \left(\omega\right)$$

$$Q_{pmn}\left(\omega\right) = \int \frac{dp_x dp_y}{\left(2\pi\right)^2} \sum_{\alpha>0,\,\beta<0} Q_{kmn}^{\alpha\beta}\left(p,\omega\right)$$

$$Q_{pmn}^{\alpha\beta}\left(p,\omega\right) = \hat{x}_k \cdot \left(v_p^{(\alpha)} - v_p^{(\beta)}\right) M_{mn}^{\alpha\beta}\left(p\right) \cdot 2\pi\delta\left(E_p^{(\alpha)} - E_p^{(\beta)} - \omega\right)$$

$$M_{mn}^{\alpha\beta}\left(p\right) = \left\langle p,\alpha\right| \Sigma_m \left|p,\beta\right\rangle \left\langle p,\alpha\right| \Sigma_n^{\dagger} \left|p,\beta\right\rangle$$

where: $\Sigma_y = \frac{1}{v_f}\left(\frac{\partial H_0}{\partial p}\right)_x$ $\Sigma_x = -\frac{1}{v_f}\left(\frac{\partial H_0}{\partial p}\right)_y$ and

$$\left(\frac{\partial H_0}{\partial p}\right)_x = \sigma^y + 3\cdot c\left(p_+^2 + p_-^2\right)\sigma^z = \begin{pmatrix} 3\cdot c\left(p_+^2+p_-^2\right) & -i \\ i & -3\cdot c\left(p_+^2+p_-^2\right) \end{pmatrix}$$

$$\left(\frac{\partial H_0}{\partial p}\right)_y = -\sigma^x + 3\cdot c\cdot i\cdot\left(p_+^2 - p_-^2\right)\sigma^z = \begin{pmatrix} 3\cdot c\cdot i\cdot\left(p_+^2-p_-^2\right) & -1 \\ -1 & -3\cdot c\cdot i\cdot\left(p_+^2-p_-^2\right) \end{pmatrix}$$

as calculated in section 1.1 the unperturbed energies satisfy: $E_{\pm p_x,\pm p_y} = E$

with the assistance of numeric calculations:

1. Q_{xyx}, Q_{xxy}, Q_{yxy}, Q_{yyx} are imaginary and different from zero
2. $Q_{xyx} = Q^{\star}_{xxy}$, $Q_{yyx} = Q^{\star}_{yxy}$
3. $Q_{yyy} = Q_{xxx} = Q_{xyy} = Q_{yxx}$

2 Simulation parameters description

The simulation computes (η_x, η_y) the current response which is described by:

$j_x = \frac{e^3 v_f^2 q\tau}{2\epsilon_0 c\hbar^2}\int_0^{\Omega}\frac{I(\omega)}{\omega^2}\eta_x(\omega)\,d\omega$

$j_y = \frac{e^3 v_f^2 q\tau}{2\epsilon_0 c\hbar^2}\int_0^{\Omega}\frac{I(\omega)}{\omega^2}\eta_y(\omega)\,d\omega$

This is done by using $2\cdot order \times 2\cdot order$ block matrix composed of 2×2 Pauli blocks. from [2] one gets:

$\lambda = 250A^3eV$

$v = 2.55eVA$

$\rightarrow c = \frac{\lambda}{2v}\frac{10^{-6}}{10^{-2}} = 0.005$

Which is a typical value for c that is used in the simulation, and magnetic field of $\bar{u} = (0.2, 0.2, 0.2)$ or $\bar{u} = (0.2, 0, 0.2)$. the results are smoothen.

The angle of all the results is zero when the current is in the x direction and 90 in the y direction

3 Simulation results

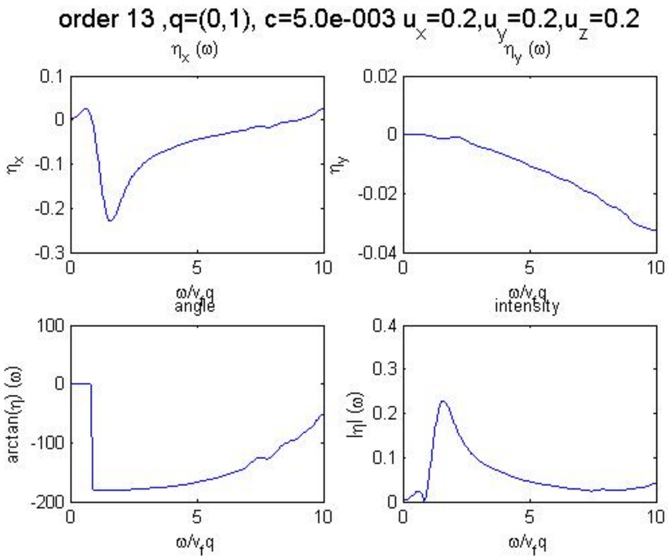


Figure 2: magnetic stripes with periodicity in y direction

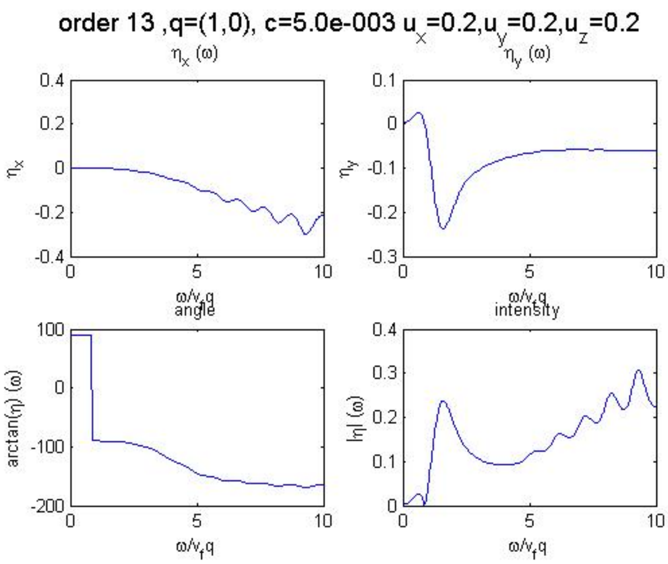


Figure 3: magnetic stripes with periodicity in x direction and high order - 13, u is zero in y direction

4 Results discussion compared to the results without C3V component

4.1 change of the current's angle with dependence in the frequency of the incoming photons:

As can be seen in figure 2 and 3. when the magnetic stripes are parallel to the y direction, the current changes direction from the y direction to the x direction with the rising of the frequency. by comparison to the results without magnetic stripes [1], the effect of the C3V symmetry on the current is greater for higher frequencies.

4.2 increase in intensity for the C3V symmetry compared to rotational symmetry:

The simulation shows an increase in $|\eta|$ at high frequencies compared to the case without the C3V symmetry.

4.3 constant direction of current when stripes are parallel to x direction.

Since x and y in this modal are not symmetrical , there is a possibility for different behavior of the current in different angles of the magnetic stripes, in contrast to placing the stripes parallel to the y direction, placing the stripes parallel to the x direction changes intensity at a much slower rate.

References

[1] Netanel H. Lindner, Gil Rafael and Felix von Oppen *Enhancement of surface photocurrents in topological insulators using magnetic superlattices* 1403.0010v1 2014.

[2] Liang Fu *Hexagonal Warping Eects in the Surface States of Topological Insulator* 1973: Random House, N.Y.

Appendix 1: The symmetries of the unperturbed Hamiltonian

1) time reversal: $TH_0T^{-1} = H_0$ with $T = i\sigma_y K$ for half spin and K is the conjugate operator. proof:

$T\breve{H}_0T^{-1} = H_0 + cT\left(p_+^3 + p_-^3\right)\sigma^z T^{-1} = H_0 + cT\left(p_+^3 + p_-^3\right)T^{-1}T\sigma^z T^{-1} =$

$= H_0 + cT\left(p_x^3 - 3p_xp_y^2 + 3p_x^2p_yi - i\cdot p_y^3 + p_x^3 - 3p_xp_y^2 - 3p_x^2p_yi + i\cdot p_y^3\right)T^{-1}T\sigma^z T^{-1}$

$= H_0 + c\left(-p_x^3 + 3p_xp_y^2 + 3p_x^2p_yi - i\cdot p_y^3 - p_x^3 + 3p_xp_y^2 - 3p_x^2p_yi + i\cdot p_y^3\right)\left(-\sigma^z\right) =$

$= H_0 + c\left(p_x^3 - 3p_xp_y^2 - 3p_x^2p_yi + i\cdot p_y^3 + p_x^3 - 3p_xp_y^2 + 3p_x^2p_yi - i\cdot p_y^3\right)\sigma^z =$

$H_0 + c\left(p_x^3 - 3p_xp_y^2 + 3p_x^2p_yi - i\cdot p_y^3 + p_x^3 - 3p_xp_y^2 - 3p_x^2p_yi + i\cdot p_y^3\right)\sigma^z = H_0 + c\left(p_+^3 + p_-^3\right)\sigma^z = \breve{H}$

2) C_3V symmetry: the Hamiltonian is symmetric in operator C_3 that transforms the state as:

$C_3\ :\ p_\pm \to e^{\pm i2\pi/3}p_\pm,\ \sigma_\pm \to e^{\pm i2\pi/3}\sigma_\pm,\ \sigma_z \to \sigma_z$

therefore: $p_x = \frac{1}{2}\left(p_+ + p_-\right) \to \frac{1}{2}\left(e^{+i2\pi/3}p_+ + e^{-i2\pi/3}p_-\right) = cos\left(\frac{2\pi}{3}\right)p_x - sin\left(\frac{2\pi}{3}\right)p_y \quad p_y = \frac{1}{2i}\left(p_+ - p_-\right) \to \frac{1}{2i}\left(e^{+i2\pi/3}p_+ - e^{-i2\pi/3}\right.$
$cos\left(\frac{2\pi}{3}\right)p_y + sin\left(\frac{2\pi}{3}\right)p_x$

$\sigma_x = \frac{1}{2}\left(\sigma_+ + \sigma_-\right) \to \frac{1}{2}\left(e^{+i2\pi/3}\sigma_+ + e^{-i2\pi/3}\sigma_-\right) = \frac{1}{2}\left(cos\left(\frac{2\pi}{3}\right)\sigma_+ + isin\left(\frac{2\pi}{3}\right)\sigma_+ + cos\left(\frac{2\pi}{3}\right)\sigma_- - isin\left(\frac{2\pi}{3}\right)\sigma_-\right) =$

$= \frac{1}{2}cos\left(\frac{2\pi}{3}\right)\left(\sigma_+ + \sigma_-\right) + \frac{1}{2}isin\left(\frac{2\pi}{3}\right)\left(\sigma_+ - \sigma_-\right) = cos\left(\frac{2\pi}{3}\right)\sigma_x - sin\left(\frac{2\pi}{3}\right)\sigma_y$

$\sigma_y = \frac{1}{2i}\left(\sigma_+ - \sigma_-\right) \to \frac{1}{2i}\left(e^{+i2\pi/3}\sigma_+ - e^{-i2\pi/3}\sigma_-\right) = \frac{1}{2i}\left(cos\left(\frac{2\pi}{3}\right)\sigma_+ + isin\left(\frac{2\pi}{3}\right)\sigma_+ - cos\left(\frac{2\pi}{3}\right)\sigma_- + isin\left(\frac{2\pi}{3}\right)\sigma_-\right) =$

$= \frac{1}{2i}cos\left(\frac{2\pi}{3}\right)\left(\sigma_+ - \sigma_-\right) + \frac{1}{2i}isin\left(\frac{2\pi}{3}\right)\left(\sigma_+ + \sigma_-\right) = cos\left(\frac{2\pi}{3}\right)\sigma_y + sin\left(\frac{2\pi}{3}\right)\sigma_x$

where: $C_3 = U_{2\pi/3} = exp\left(i\frac{2\pi}{3}n\left(\sigma^z/2 + L_z/\hbar\right)\right)$

and also under the M: transformation:

$M\ :\ k_+ \leftrightarrow -k_-,\ \sigma_x \leftrightarrow \sigma_x,\ \sigma_{y,z} \leftrightarrow -\sigma_{y,z}$

$M\ :\ k_x = \frac{1}{2}\left(k_+ + k_-\right) \to \frac{1}{2}\left(-k_- - k_+\right) = -k_x, \quad k_y = \frac{1}{2}\left(k_+ - k_-\right) \to \frac{1}{2}\left(-k_- + k_+\right) = k_y$

where M is: $M = \Pi_x\sigma_x$

proof: $\Pi_x\sigma_x\sigma_y\sigma_x^{-1}\Pi_x^{-1} = i\sigma_z\sigma_x^{-1} = i\sigma_z\sigma_x = i^2\sigma_y = -\sigma_y$

$\Pi_x\sigma_x\sigma_z\sigma_x^{-1}\Pi_x^{-1} = -i\sigma_y\sigma_x^{-1} = -i\sigma_y\sigma_x = i^2\sigma_z = -\sigma_z$

$\Pi_x\sigma_x\sigma_x\sigma_x^{-1}\Pi_x^{-1} = \Pi_x\sigma_x\Pi_x^{-1} = \sigma_x$

Proof for Pauli operator:

$\sigma_z\sigma_\pm = \sigma_z\left(\sigma_x \pm i\sigma_y\right) = i\sigma_y \pm i\cdot(-i\sigma_x) = \pm\sigma_x + i\sigma_y = \pm\sigma_\pm$

$\sigma_\pm\sigma_z = \left(\sigma_x \pm i\sigma_y\right)\sigma_z = -i\sigma_y \mp i\cdot(i\sigma_x) = \mp\sigma_x - i\sigma_y = \mp\sigma_\pm$

$exp\left(i\frac{2\pi}{3}n\left(\sigma^z/2\right)\right)\sigma_+ exp\left(-i\frac{2\pi}{3}n\left(\sigma^z/2\right)\right) = \left(cos\left(\frac{\pi}{3}\right) + i\sigma_z sin\left(\frac{\pi}{3}\right)\right)\sigma_+\left(cos\left(\frac{\pi}{3}\right) - i\sigma_z sin\left(\frac{\pi}{3}\right)\right) = \frac{1}{4}\sigma_+ + \frac{\sqrt{3}}{4}i\cdot\sigma_+ + \frac{\sqrt{3}}{4}i\cdot\sigma_+ - \frac{3}{4}\sigma_+ = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sigma_+ = e^{i2\pi/3}\sigma_+$

$exp\left(i\frac{2\pi}{3}n\left(\sigma^z/2\right)\right)\sigma_- exp\left(-i\frac{2\pi}{3}n\left(\sigma^z/2\right)\right) = \left(cos\left(\frac{\pi}{3}\right) + i\sigma_z sin\left(\frac{\pi}{3}\right)\right)\sigma_-\left(cos\left(\frac{\pi}{3}\right) - i\sigma_z sin\left(\frac{\pi}{3}\right)\right) = \frac{1}{4}\sigma_- - \frac{\sqrt{3}}{4}i\cdot\sigma_- - \frac{\sqrt{3}}{4}i\cdot\sigma_- - \frac{3}{4}\sigma_- = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sigma_- = e^{-i2\pi/3}\sigma_-$

$exp\left(i\frac{2\pi}{3}n\left(\sigma^z/2\right)\right)\sigma_x exp\left(-i\frac{2\pi}{3}n\left(\sigma^z/2\right)\right) = \left(cos\left(\frac{\pi}{3}\right) + i\sigma_z sin\left(\frac{\pi}{3}\right)\right)\sigma_x\left(cos\left(\frac{\pi}{3}\right) - i\sigma_z sin\left(\frac{\pi}{3}\right)\right) = \frac{1}{4}\sigma_x + i\left(i\sigma_y\right)\frac{\sqrt{3}}{2} - \frac{3}{4}\sigma_x = -\frac{1}{2}\sigma_x - \frac{\sqrt{3}}{2}\sigma_y.$

The proof that the Hamiltonian is symmetric by these operators:

$C_3\breve{H}_0C_3^{-1} = H_0 + cC_3\left(p_+^3 + p_-^3\right)\sigma^z C_3^{-1} = H_0 + cC_3\left(p_+^3 + p_-^3\right)C_3^{-1}\sigma^z = H_0 + c\left(e^{+i3\cdot 2\pi/3}p_+^3 + e^{-i3\cdot 2\pi/3}p_-^3\right)\sigma^z = \breve{H}_0$

$M\breve{H}_0M^{-1} = Mv_f\left(p_x\sigma^y - p_y\sigma^x\right)M^{-1} + cM\left(p_+^3 + p_-^3\right)\sigma^z M^{-1} = v_f\left(\left(-p_x\right)\left(-\sigma^y\right) - p_y\sigma^x\right) - c\left(p_+^3 + p_-^3\right)\left(-\sigma^z\right) = \breve{H}_0$

therefore $E_{k_x,k_y} = E_{cos\left(\frac{2\pi}{3}\right)k_x - sin\left(\frac{2\pi}{3}\right)k_y . cos\left(\frac{2\pi}{3}\right)k_y + sin\left(\frac{2\pi}{3}\right)k_x}$

$E_{k_x,k_y} = E_{-k_x,k_y}$

3) Mirror of y axes: $M_y = \Pi_y\sigma_y$

$\sigma_y\sigma_z\sigma_y^{-1} = i\sigma_x\sigma_y^{-1} = i\sigma_x\sigma_y = i^2\sigma_z = -\sigma_z$

$\sigma_y\sigma_x\sigma_y^{-1} = -\sigma_x$

$M_y\breve{H}_0M_y^{-1} = M_yv_f\left(p_x\sigma^y - p_y\sigma^x\right)M_y^{-1} + M_yc\left(p_+^3 + p_-^3\right)\sigma^z M_y^{-1} = v_f\left(p_x\sigma^y - p_y\sigma^x\right) - c\left(p_+^3 + p_-^3\right)\sigma^z$

In this case the Eigen energies remain the same but not the Eigen vectors

Symmetries of the Hamiltonian with magnetic stripes

Azimuthal symmetry: obviously doesn't exist.

Time symmetry remains with a new operator $\tilde{T} = TM$ with $MxM^\dagger = y + \frac{\pi}{q}$

or in matrix form $M_{ij} = \delta_{ij}\frac{j+\pi/q}{j} \rightarrow M_{ij}^\dagger = \delta_{i-j}\frac{-j+\pi/q}{-j} = M_{ij}^{-1}$ therefore M is unitary and $\tilde{T}$ is a anti-unitary.

the M transformation only revert the magnetic field and therefore complete $\tilde{T}$ to be time reversal operator for this Hamiltonian.

proof of symmetry: $\tilde{T}\left(\breve{H}_0 + V\right)\tilde{T}^\dagger = TM\cdot\breve{H}_0\cdot M^\dagger T^\dagger + TM\cdot V\cdot M^\dagger T^\dagger = \breve{H}_0 + T\left(-V\right)T^\dagger = \breve{H}_0 + V$

the effect on the eigenstates is: $T\left|k_x,k_y;\alpha\right\rangle = \left|-k_x,-k_y;\alpha\right\rangle$ since T is a time reversal operator. Because of symmetry of the Hamiltonian the Eigen values stay the same i.e $E_{-k_x,-k_y,\alpha} = E_{k_x,k_y,\alpha}$.

Particle hole symmetry: in this system there is a particle-hole symmetry. it is the symmetry under the change of the sign of charges. the operator is $C = \Pi_x\Pi_yT$ where $\Pi_x x\Pi_x^\dagger = -x$ is a reflection operator (therefor it reflects the magnetic field)

proof: $\breve{H}C\left|\psi\right\rangle = \alpha C\left|\psi\right\rangle \rightarrow C^{-1}\left(\breve{H}_0 + V\right)C\left|\psi\right\rangle = C^{-1}\left(H + V\right)C\left|\psi\right\rangle = -H + cT^{-1}\Pi_y^{-1}\Pi_x^{-1}\left(p_+^3 + p_-^3\right)\sigma^z\Pi_x\Pi_yT = -H - cT^{-1}\left(p_+^3 + p_-^3\right)\sigma^zT = \alpha\left|\psi\right\rangle$

$= -H - c\left(p_+^3 + p_-^3\right)\sigma^z = \alpha\left|\psi\right\rangle \rightarrow -\left(\breve{H}_0 + V\right)\left|\psi\right\rangle = \alpha\left|\psi\right\rangle \rightarrow \breve{H}\left|\psi\right\rangle = -\alpha\left|\psi\right\rangle \rightarrow C\left|\alpha\right\rangle = \left|-\alpha\right\rangle$

since T reverse momentum, but so does Π. C doesn't have a net affect on momentum. therefore: $C\left|k_x,k_y;\alpha\right\rangle = \left|k_x,k_y;-\alpha\right\rangle$ and of course $E_{k_x,k_y;\alpha} = -E_{k_x,k_y;-\alpha}$

Appendix 2

Calculation of the current response

The current of electron which were excited by photons and are in the band α is $j = e\int \frac{d^2k}{(2\pi)^2}\sum_\alpha\left[v_{\alpha,k}\left(n_{k,\alpha}-n^0_{k,\alpha}\right)\right]$ where $\frac{d^2k}{(2\pi)^2}$ is density of states in 2d k space. $n_{k,\alpha}$is the distribution function induced by the incident light, $n^0_{k,\alpha}$is the equilibrium distribution function. $v_{\alpha,k}$ is the velocity of the particle with k momentum in α band.

withing the relaxation time approximation the number of particles excited by light in the band $\alpha > 0$ is $\left(n_{k,\alpha}-n^0_{k,\alpha}\right) = \tau\sum_{\beta<0}\Gamma\left(k,\beta\to k,\alpha\right)\left(n^0_{k,\beta}-n^0_{k,\alpha}\right)$

where Γ is the probability rate of changing state from α to β band both states with momentum k (probably because photon have little momentum), where

$\tau\Gamma\left(k,\beta\to k,\alpha\right)n^0_{k,\beta}$ is the number of particles moving from β to α. and $\tau\Gamma\left(k,\beta\to k,\alpha\right)n^0_{k,\alpha}$ is the number of particles moving from α to β. for $T=0$:

$j = e\int \frac{d^2k}{(2\pi)^2}\sum_\alpha\left[v_{\alpha,k}\left(n_{k,\alpha}-n^0_{k,\alpha}\right)\right]$

$j = e\int \frac{d^2k}{(2\pi)^2}\sum_{\alpha>0}\left[v_{\alpha,k}\left(n_{k,\alpha}-n^0_{k,\alpha}\right)\right] + e\int \frac{d^2k}{(2\pi)^2}\sum_{\alpha<0}\left[v_{\alpha,k}\left(n_{k,\alpha}-n^0_{k,\alpha}\right)\right] =$

$j = e\int \frac{d^2k}{(2\pi)^2}\sum_{\alpha>0}\left[v_{\alpha,k}\left(n_{k,\alpha}-n^0_{k,\alpha}\right)\right] + e\int \frac{d^2k}{(2\pi)^2}\sum_{\beta<0}\left[v_{\beta,k}\left(n_{k,\beta}-n^0_{k,\beta}\right)\right] =$

$j = e\tau\int \frac{d^2k}{(2\pi)^2}\sum_{\alpha>0,\beta<0}v_{\alpha,k}\Gamma\left(k,\beta\to k,\alpha\right)\left(n^0_{k,\beta}-n^0_{k,\alpha}\right) - e\tau\int \frac{d^2k}{(2\pi)^2}\sum_{\alpha>0,\beta<0}v_{\beta,k}\Gamma\left(k,\beta\to k,\alpha\right)\left(n^0_{k,\beta}-n^0_{k,\alpha}\right) =$

$j = e\tau\int \frac{d^2k}{(2\pi)^2}\sum_{\alpha>0,\beta<0}\left(v_{\alpha,k}-v_{\beta,k}\right)\Gamma\left(k,\beta\to k,\alpha\right)\left(n^0_{k,\beta}-n^0_{k,\alpha}\right) \overset{T=0\to n^0_{k,\beta}-n^0_{k,\alpha}=1}{=}$

$= e\tau\int \frac{d^2k}{(2\pi)^2}\sum_{\alpha>0,\beta<0}\left(v_{\alpha,k}-v_{\beta,k}\right)\Gamma\left(k,\beta\to k,\alpha\right)$

the transition rates are: $\Gamma\left(k,\beta\to k,\alpha\right) = |\left\langle k,\alpha\right|H_{int}\left(\omega\right)\left|k,\beta\right\rangle|^2\frac{2\pi}{\hbar}\delta\left(E_{k,\alpha}-E_{k,\beta}-\omega\right)$

proof: using Fermi golden rule that is: $\Gamma \simeq \frac{2\pi}{\hbar}\sum_E\rho\left(E\right)|V_{Ei}|^2\delta\left(E-E_i-\hbar\omega\right)$ the results are obtained directly.

Interaction Hamiltonian: $\hat{H}_{int} = e\frac{\partial H_0}{\partial p}\cdot A\left(x,t\right)$ where A is the vector potential and $\frac{\partial H_0}{\partial p} = \frac{\partial r}{\partial t}$ according to Hamilton equations.

Incoming light circularly polarized: $E\left(t\right) = E_c\left(\hat{x}cos\left(\omega t\right)+\hat{y}sin\left(\omega t\right)\right) \to A\left(\omega\right) = \frac{E_c}{\omega}\left(\hat{x}sin\left(\omega t\right)-\hat{y}\cdot cos\left(\omega t\right)\right) = \frac{1}{2i}\frac{E_c}{\omega}\left(\left(\hat{x}-i\hat{y}\right)e^{i\omega t}-\left(\hat{x}+i\hat{y}\right)e^{-i\omega t}\right)$

The current response: $j_k = \frac{e\tau}{\hbar}\left(ev_f\right)^2\left(\frac{1}{\omega^2}\right)E_m\left(\omega\right)Q_{kmn}\left(\omega\right)E_n*\left(\omega\right)$ where $E_x\left(\pm\omega\right) = E_c/2, \quad E_y\left(\omega\right)\pm E_c/2i$ where $k,m,n\in x,y$ and indexes appearing twice are summed over i.e m,n.

$$j = e\tau\int\frac{d^2k}{\left(2\pi\right)^2}\sum_{\alpha>0,\beta<0}\left[\left(v_{\alpha,k}-v_{\beta,k}\right)\left\langle k,\alpha\right|H_{int}\left(\omega\right)\left|k,\beta\right\rangle|^2\frac{2\pi}{\hbar}\delta\left(E_{k,\alpha}-E_{k,\beta}-\omega\right)\right] =$$

Substituting H_{int} :

$$j = \frac{e\tau}{\hbar}\int\frac{d^2k}{\left(2\pi\right)^2}\sum_{\alpha>0,\beta<0}\left[\left(v_{\alpha,k}-v_{\beta,k}\right)\left\langle k,\alpha\right|e\frac{\partial H_0}{\partial p}\cdot A\left(\omega\right)\left|k,\beta\right\rangle|^2 2\pi\delta\left(E_{k,\alpha}-E_{k,\beta}-\omega\right)\right] =$$

Substituting $A\left(\omega\right)$:

$$j = \frac{e\tau}{\hbar}\int\frac{d^2k}{\left(2\pi\right)^2}\sum_{\alpha>0,\beta<0}\left[\left(v_{\alpha,k}-v_{\beta,k}\right)\left|\left\langle k,\alpha\right|e\frac{1}{2i}\frac{E_c}{\omega}\frac{\partial H_0}{\partial p}\cdot\left(\left(\hat{x}-i\hat{y}\right)e^{i\omega t}-\left(\hat{x}+i\hat{y}\right)e^{-i\omega t}\right)\left|k,\beta\right\rangle\right|^2 2\pi\delta\left(E_{k,\alpha}-E_{k,\beta}-\omega\right)\right] =$$

$$j = \frac{e\tau}{\hbar}\frac{1}{\omega^2}\int\frac{d^2k}{(2\pi)^2}\sum_{\alpha>0,\beta<0}\left[\left(v_{\alpha,k}-v_{\beta,k}\right)\left|\left\langle k,\alpha\right|e\frac{1}{2i}\frac{E_c}{\omega}\left\{\left(\frac{\partial H_0}{\partial p}\right)_x\left(e^{i\omega t}-e^{-i\omega t}\right)-i\cdot\left(\frac{\partial H_0}{\partial p}\right)_y\left(e^{i\omega t}+e^{-i\omega t}\right)\right\}\left|k,\beta\right\rangle\right|^2 =\right.$$

$\cdot 2\pi\delta\left(E_{k,\alpha}-E_{k,\beta}-\omega\right)\Big]$

$E\left(t\right)=E_c\left(\hat{x}cos\left(\omega t\right)+\tilde{y}sin\left(\omega t\right)\right)=E_c\left(\hat{x}\frac{1}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)+\tilde{y}\frac{1}{2\cdot i}\left(e^{i\omega t}-e^{-i\omega t}\right)\right)=\left(E_x\hat{x}+E_y\hat{y}\right)$

From the unperturbed Hamiltonian: $\left(\frac{\partial H_0}{\partial p}\right)_x=\Sigma_y$ $\left(\frac{\partial H_0}{\partial p}\right)_y=-\Sigma_x$ substituting that and the identeties for E_c:$E_x\left(\pm\omega\right)=\frac{E_c}{2}, E_y\left(\omega\right)=\frac{E_c}{2i}$

$$j=\frac{e\tau}{\hbar}\frac{1}{\omega^2}\int\frac{d^2k}{\left(2\pi\right)^2}\sum_{\alpha>0,\beta<0}\left[\left(v_{\alpha,k}-v_{\beta,k}\right)\left|\left\langle k,\alpha\right|e\left\{\Sigma_yE_y+E_x\cdot\Sigma_x\right\}\left|k,\beta\right\rangle\right|^2 2\pi\delta\left(E_{k,\alpha}-E_{k,\beta}-\omega\right)\right]\rightarrow$$

$$j=\frac{e\tau}{\hbar}\frac{1}{\omega^2}\int\frac{d^2k}{\left(2\pi\right)^2}\sum_{\alpha>0,\beta<0}\left[\left(v_{\alpha,k}-v_{\beta,k}\right)e^2v_F^2\sum_{m,n}E_m\left\langle k,\alpha\right|\Sigma_m\left|k,\beta\right\rangle\left\langle k,\beta\right|\Sigma_n^\dagger\left|k,\alpha\right\rangle E_n^\star 2\pi\delta\left(E_{k,\alpha}-E_{k,\beta}-\omega\right)\right]=$$

Define $M_{mn}^{\alpha\beta}\left(k\right)=\left\langle k,\alpha\right|\Sigma_m\left|k,\beta\right\rangle\left\langle k,\alpha\right|\Sigma_n^\dagger\left|k,\beta\right\rangle$ therefore:

$$j=\frac{e\tau}{\hbar}\frac{1}{\omega^2}\int\frac{d^2k}{\left(2\pi\right)^2}\sum_{\alpha>0,\beta<0}\left[\left(v_{\alpha,k}-v_{\beta,k}\right)e^2v_F^2\sum_{m,n}E_mM_{mn}^{\alpha\beta}\left(k\right)E_n^\star 2\pi\delta\left(E_{k,\alpha}-E_{k,\beta}-\omega\right)\right]=$$

Define $Q_{kmn}^{\alpha\beta}\left(\omega\right)=\hat{x}_k\cdot\left(v_{\mathbf{k}}^{(\alpha)}-v_{\mathbf{k}}^{(\beta)}\right)M_{mn}^{\alpha\beta}\left(\mathbf{k}\right)\times 2\pi\delta\left(E_{\mathbf{k},\alpha}-E_{\mathbf{k},\beta}-\omega\right)$ therefore: (notice k and $\mathbf{k}$ are different)

$$j_k=\frac{e\tau}{\hbar}\frac{1}{\omega^2}\int\frac{d^2\mathbf{k}}{\left(2\pi\right)^2}\sum_{\alpha>0,\beta<0}\left[e^2v_F^2\sum_{m,n}E_mQ_{kmn}^{\alpha\beta}E_n^\star\right]=$$

Define $Q_{kmn}\left(\omega\right)=\int\frac{dk_xdk_y}{\left(2\pi\right)^2}\sum_{\alpha>0,\beta<0}Q_{kmn}^{\alpha\beta}\left(k,\omega\right)$ therefore:

$$j_k=\frac{e\tau}{\hbar}\frac{1}{\omega^2}e^2v_F^2\sum_{m,n}E_mQ_{kmn}E_n^\star=\frac{e\tau}{\hbar}\left(ev_f\right)^2\left(\frac{1}{\omega^2}\right)E_m\left(\omega\right)Q_{kmn}\left(\omega\right)E_n*\left(\omega\right)$$